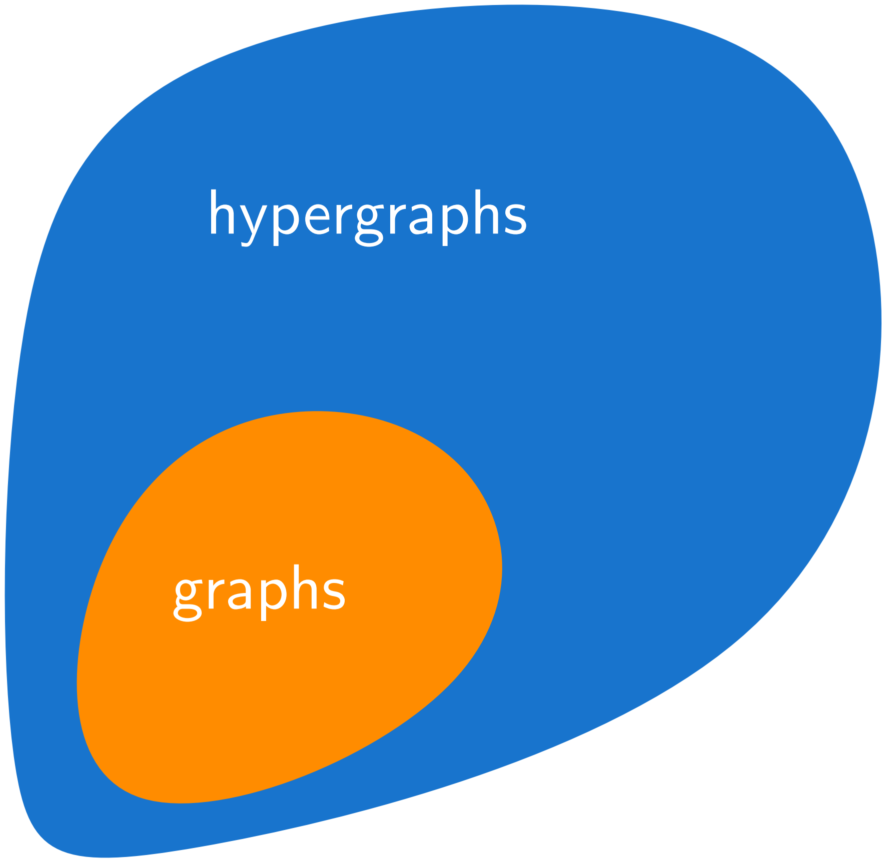

Drawing Graphs and Hypergraphs in 2D & 3D

Alexander Wolff @ ICCG 2020

What is graph drawing?

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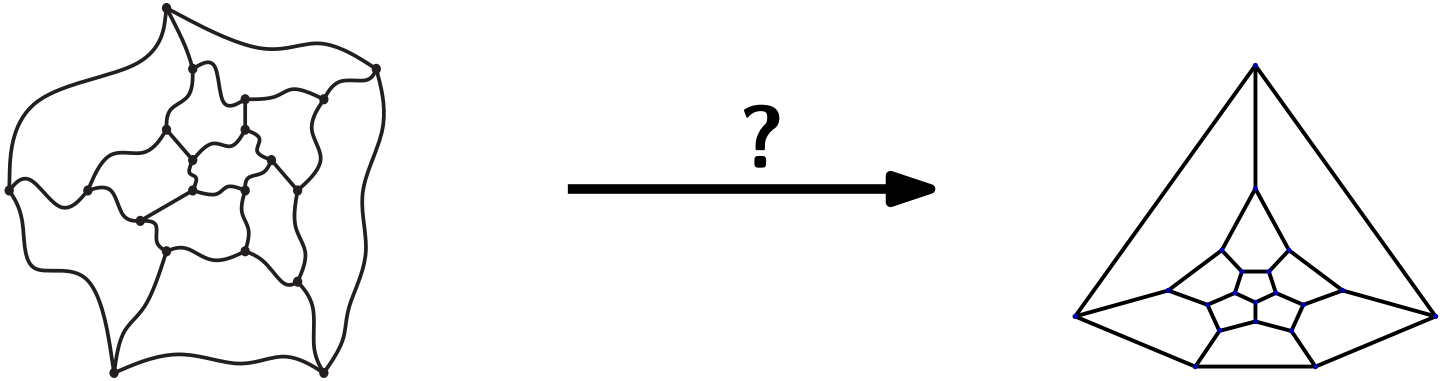

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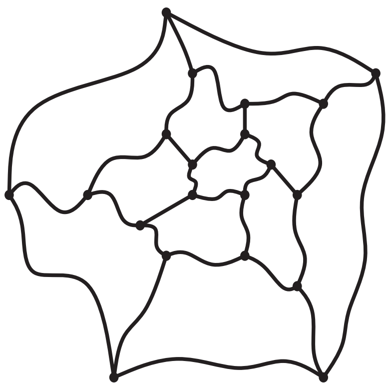

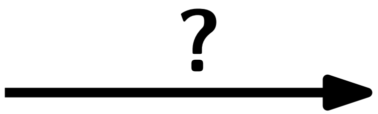

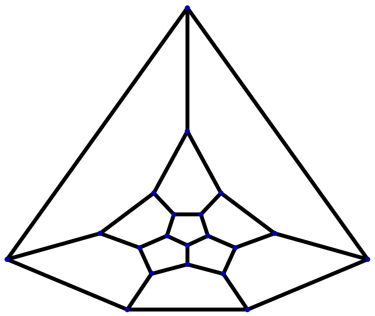

- abstract (combinatorial) graph

- drawing (e.g. node-link diagram)

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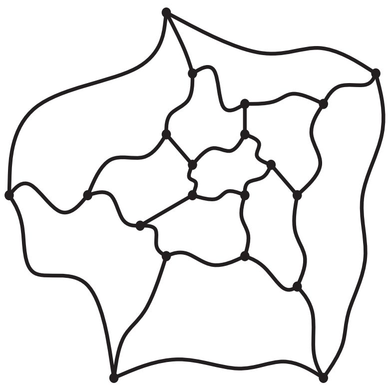
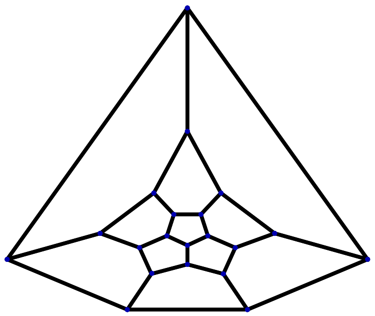


?

ALGORITHM



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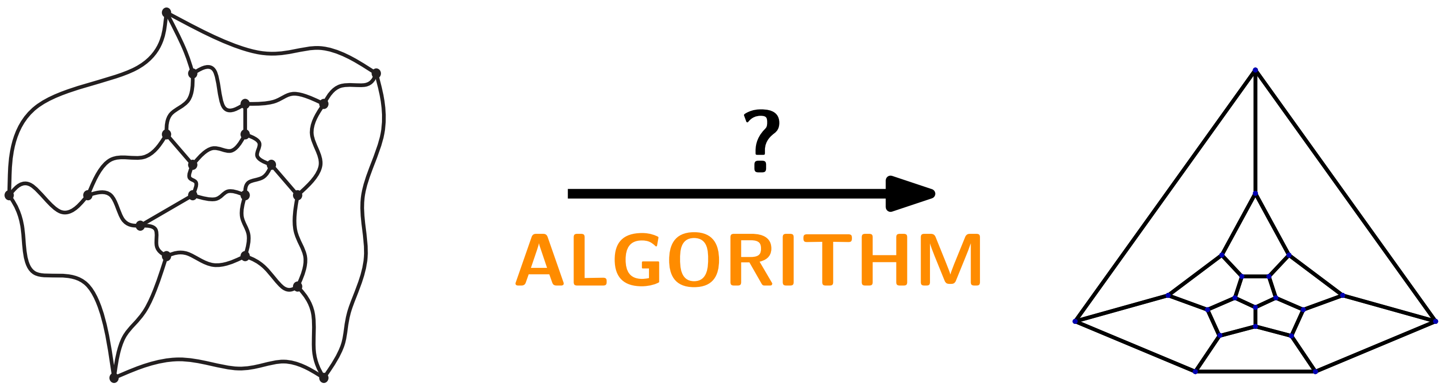

- abstract (combinatorial) graph

- drawing (e.g. node-link diagram)

Goal: Algorithm guarantees a (provable) geometric quality measure in the worst case

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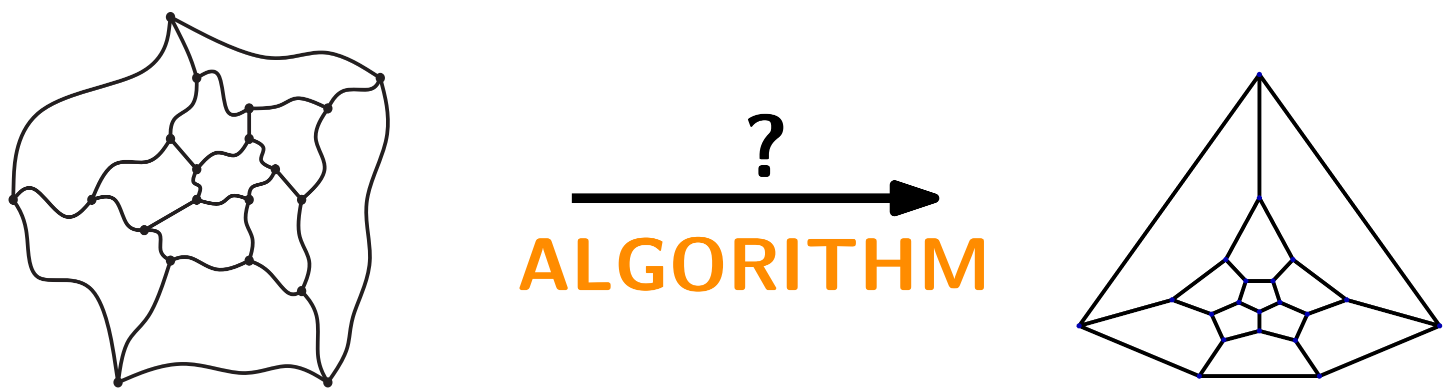


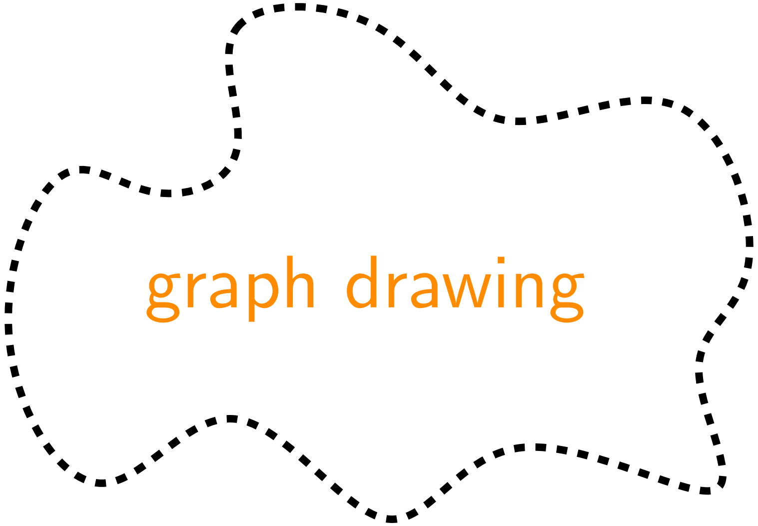
- abstract (combinatorial) graph

- drawing (e.g. node-link diagram)

Goal: Algorithm guarantees a (provable) geometric quality measure in the worst case

Evaluation is not task-driven

The many dimensions of graph drawing



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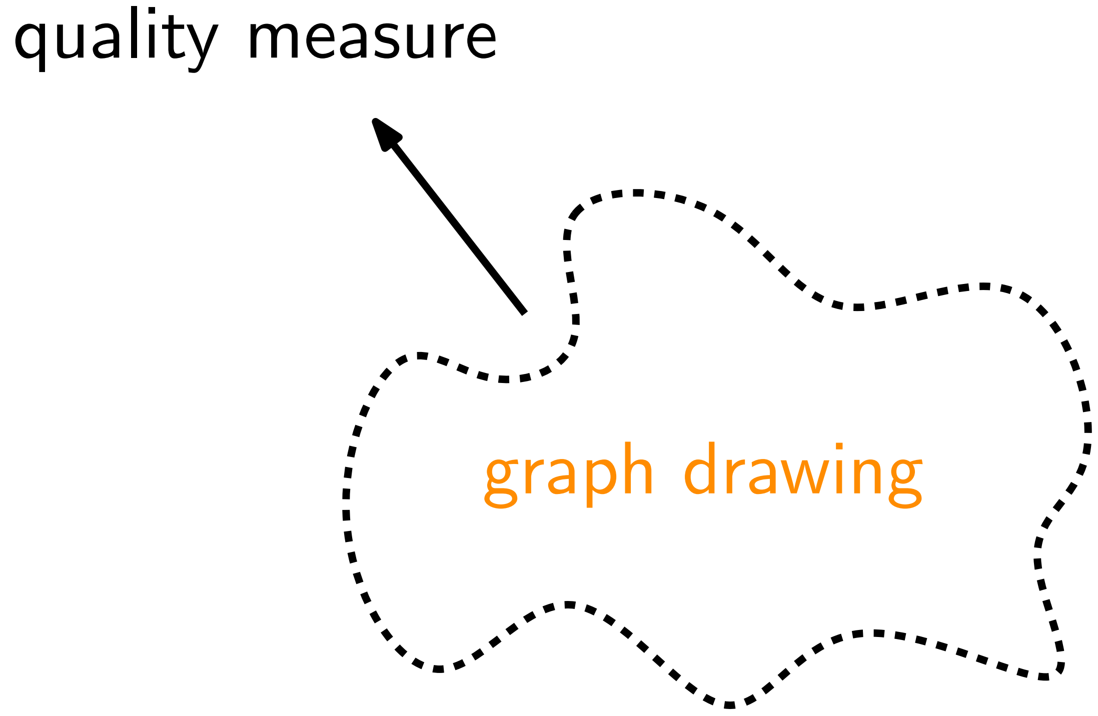

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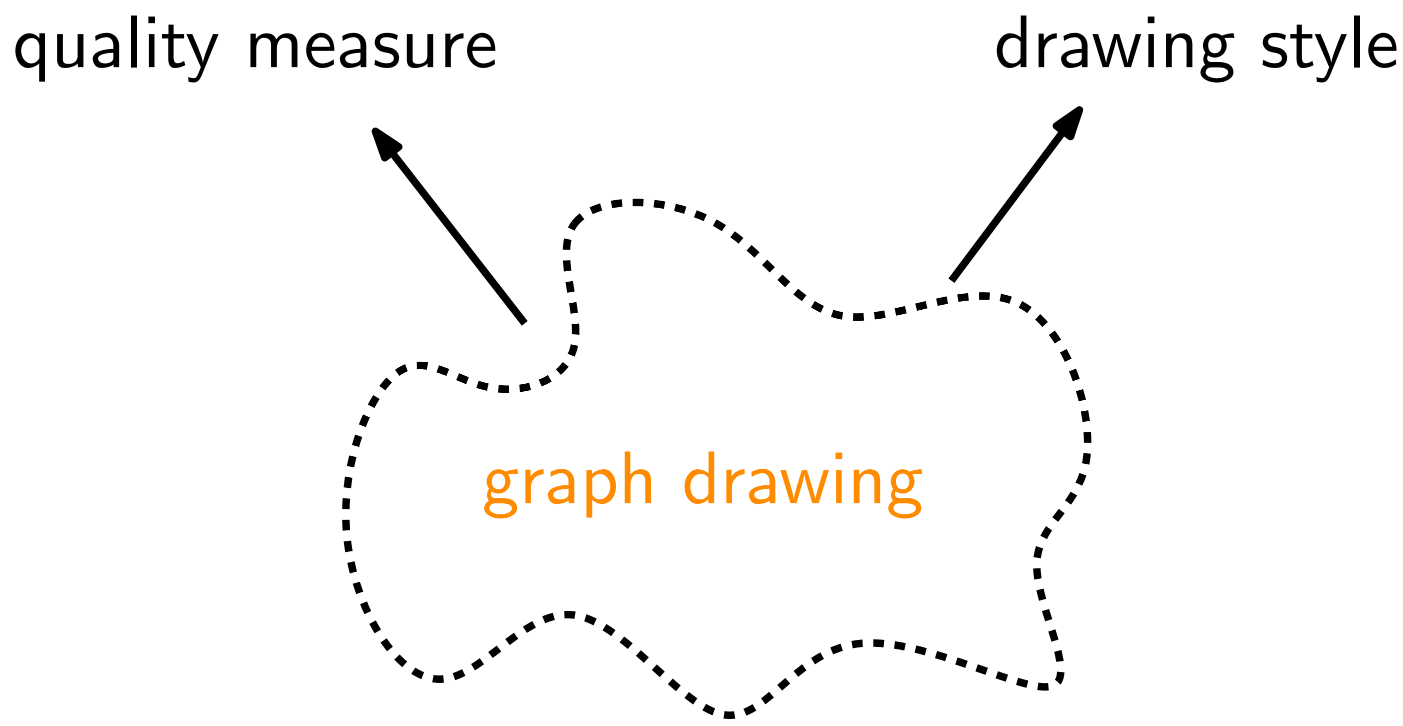

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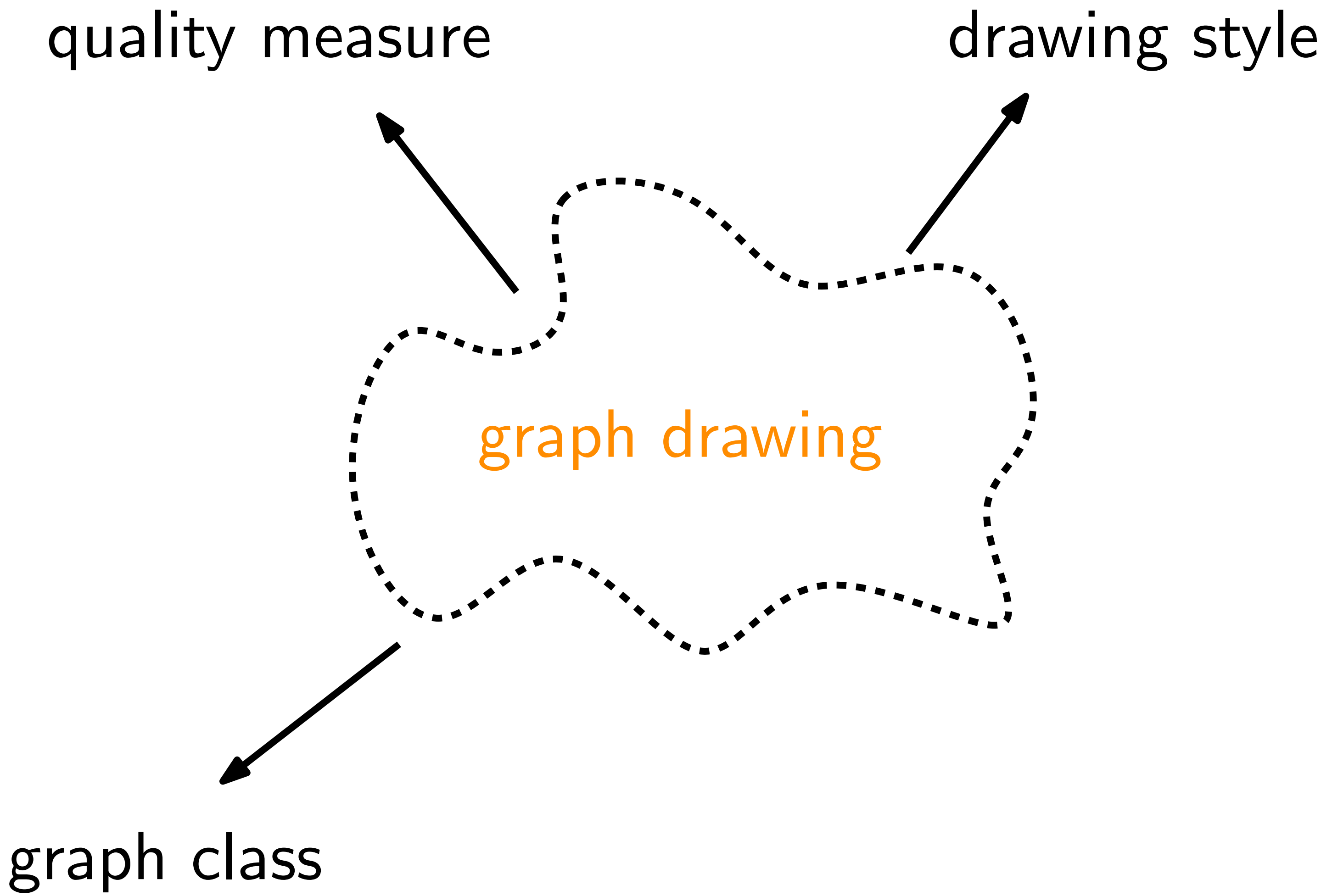

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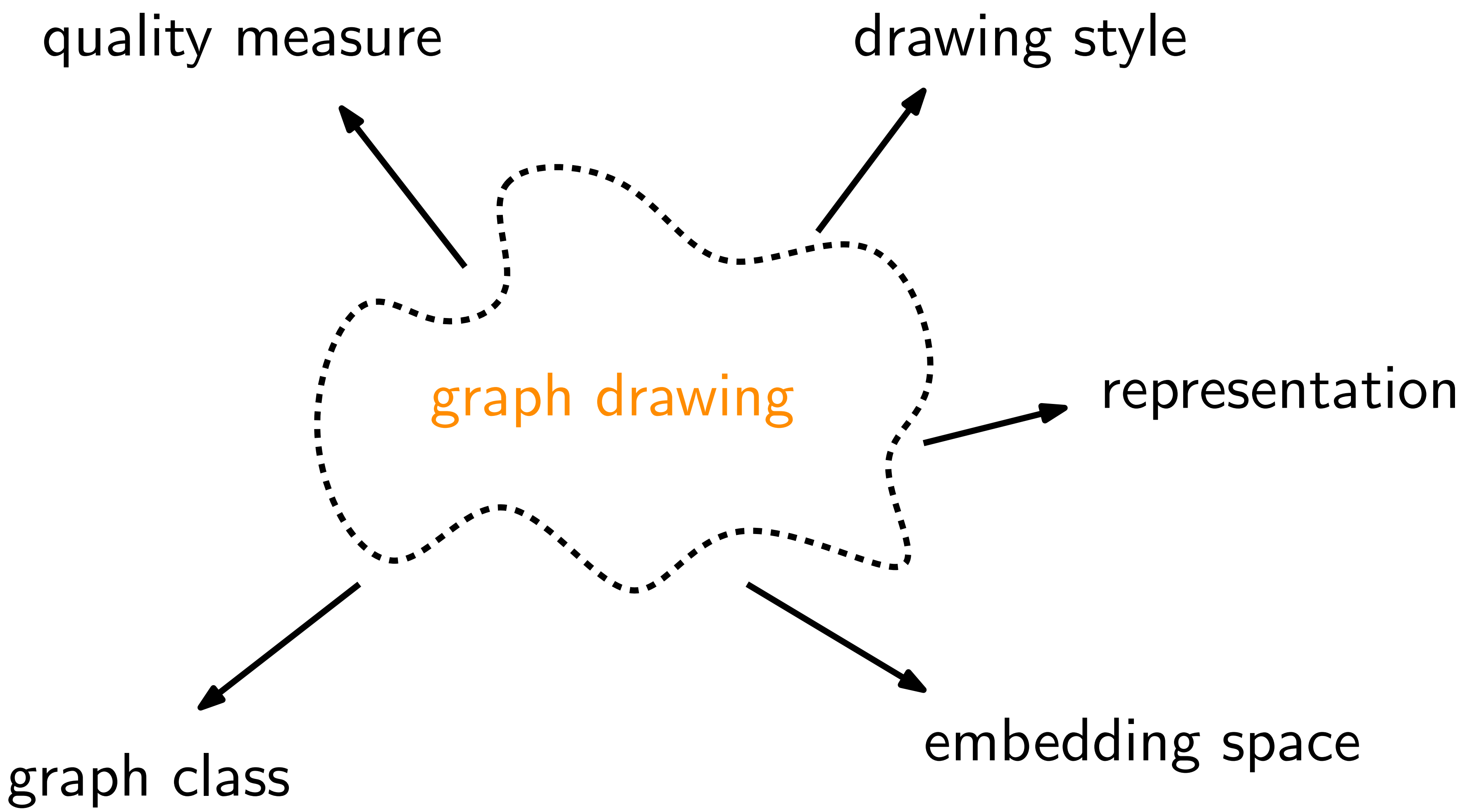

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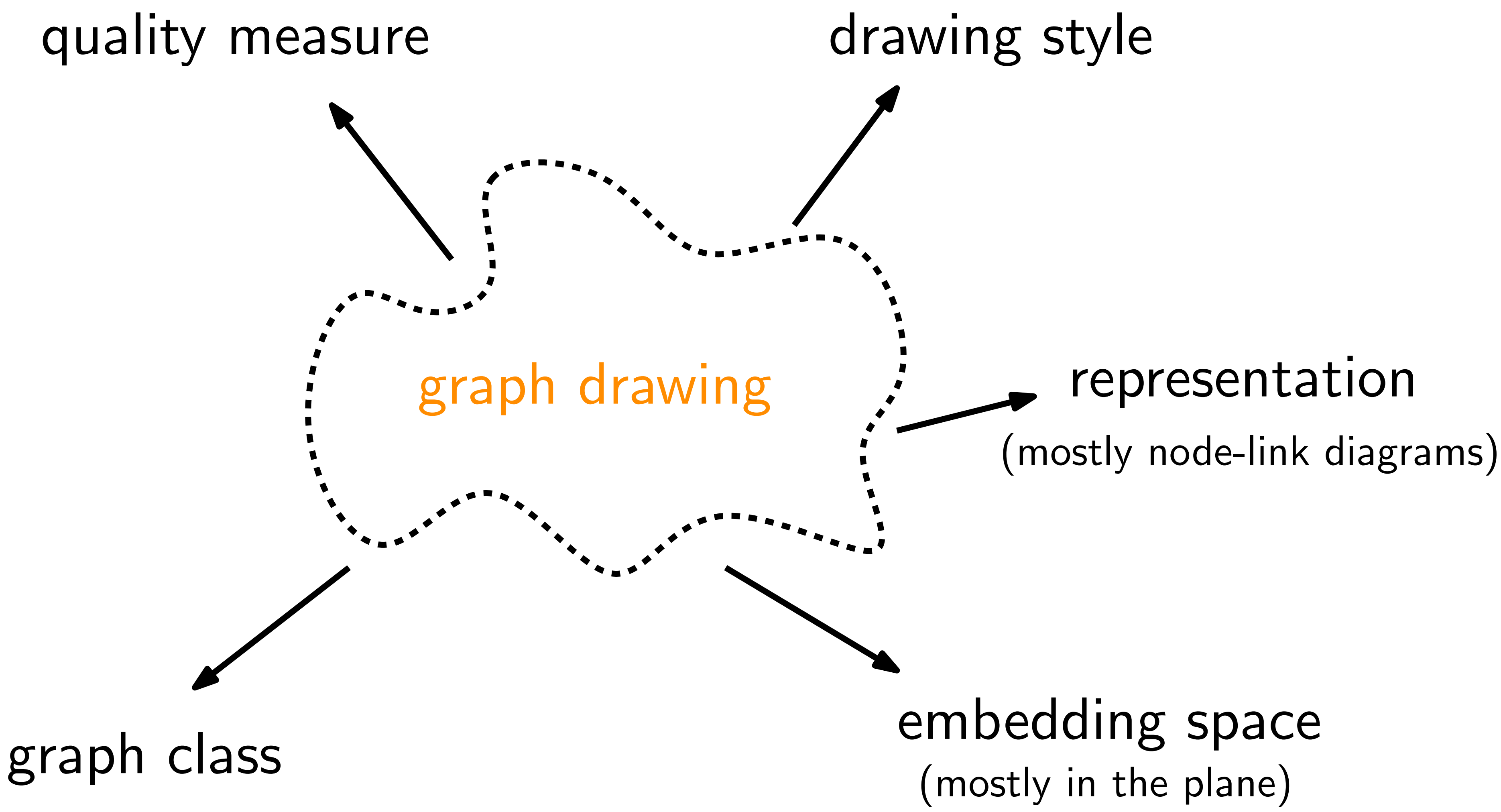

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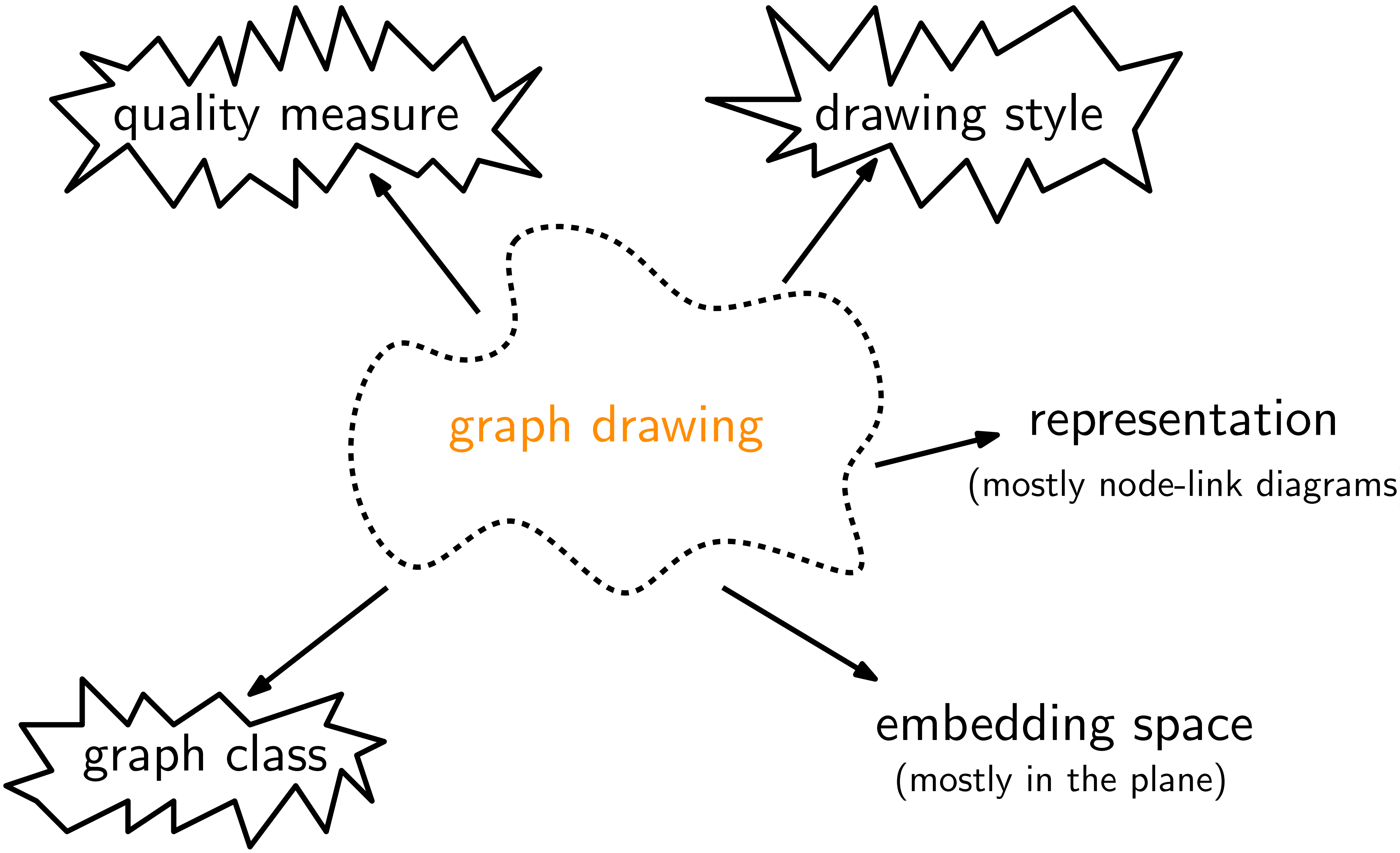

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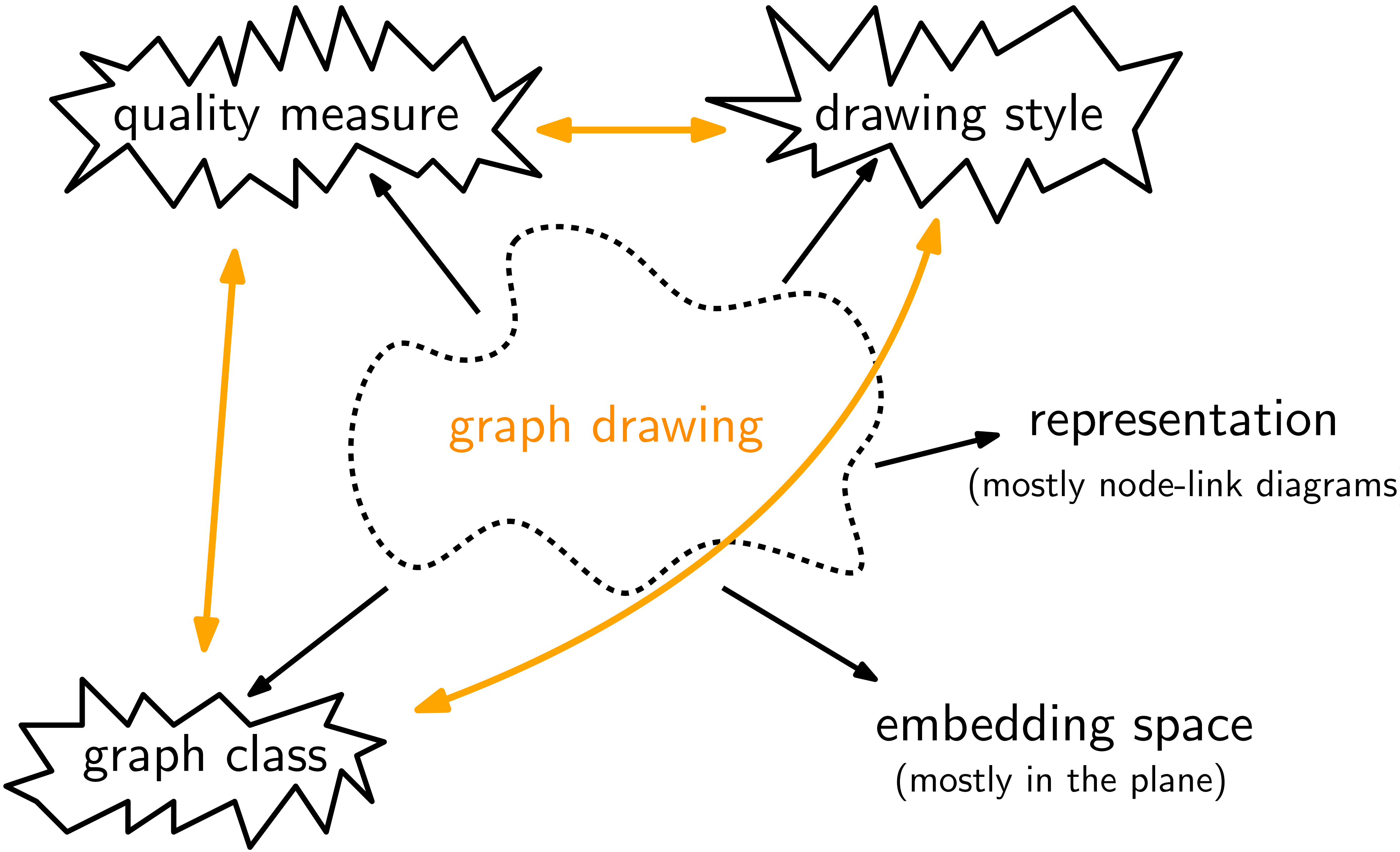

Drawing Styles

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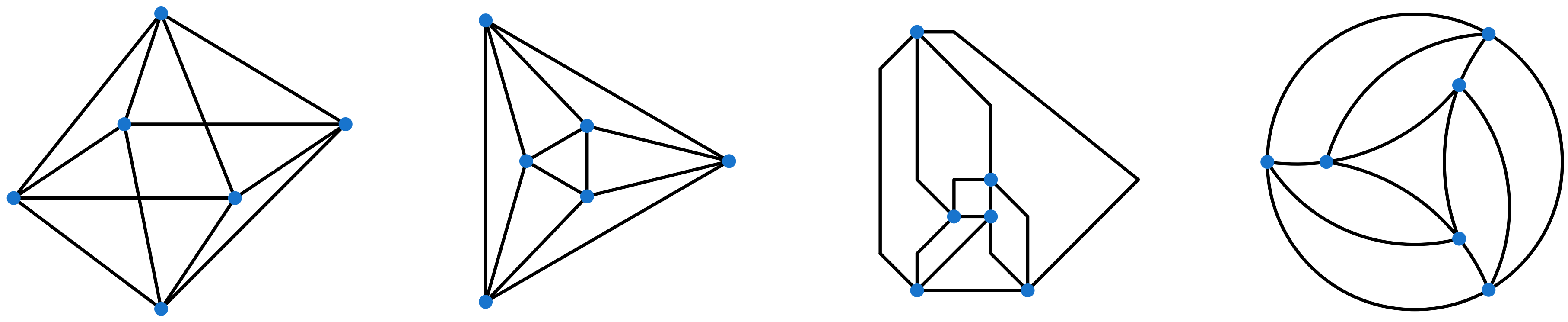

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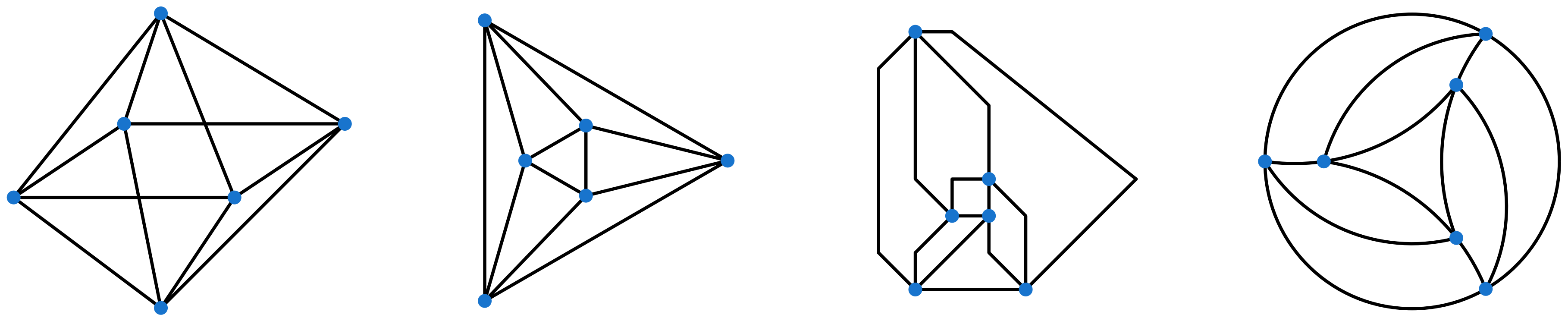
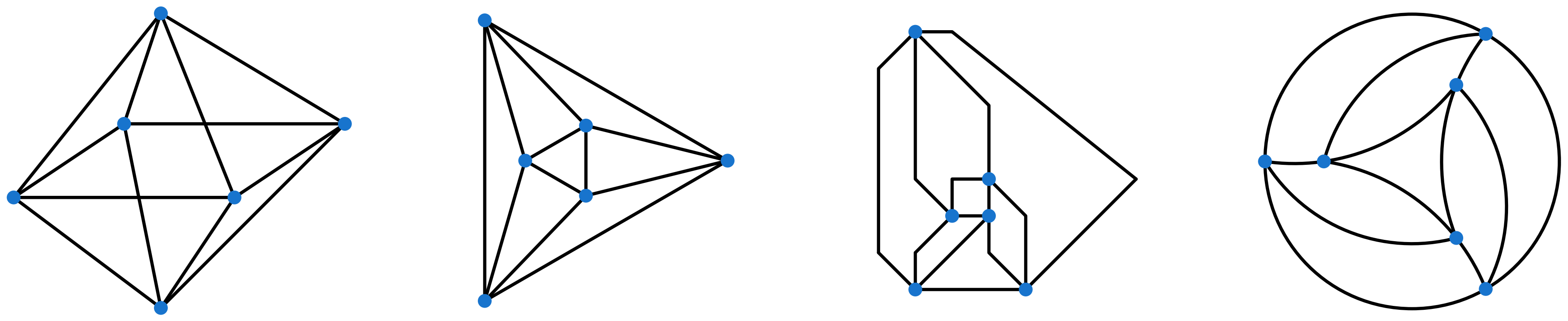


- straight-line vs. curved

Drawing Styles



- straight-line vs. curved
- straight-line vs. polyline

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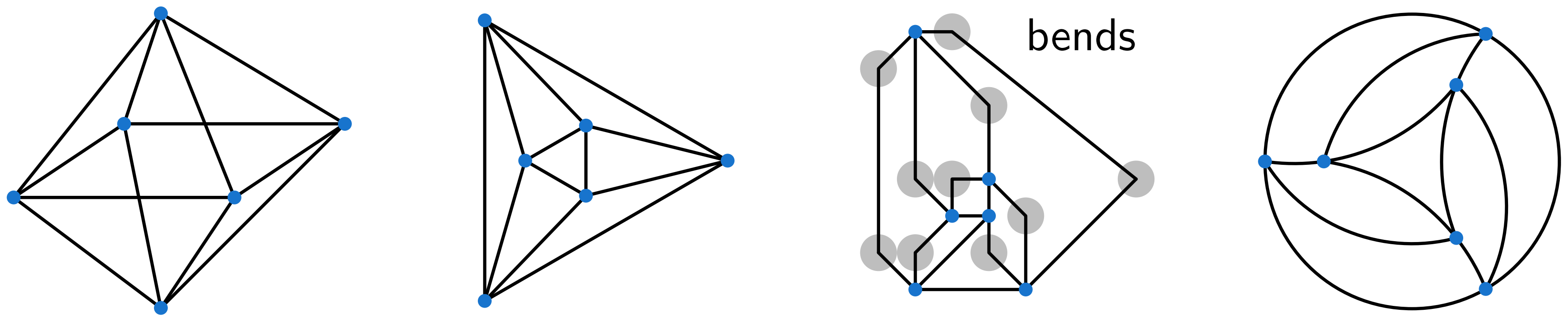

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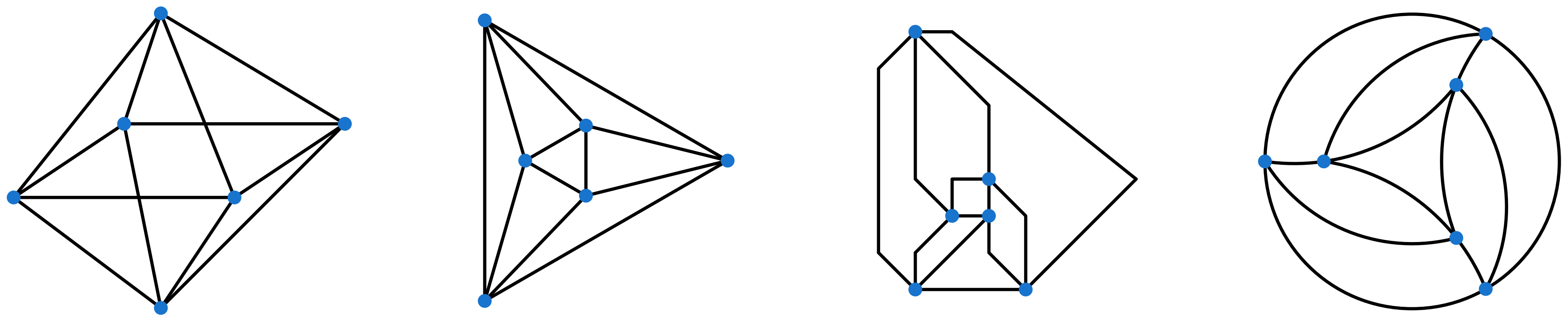

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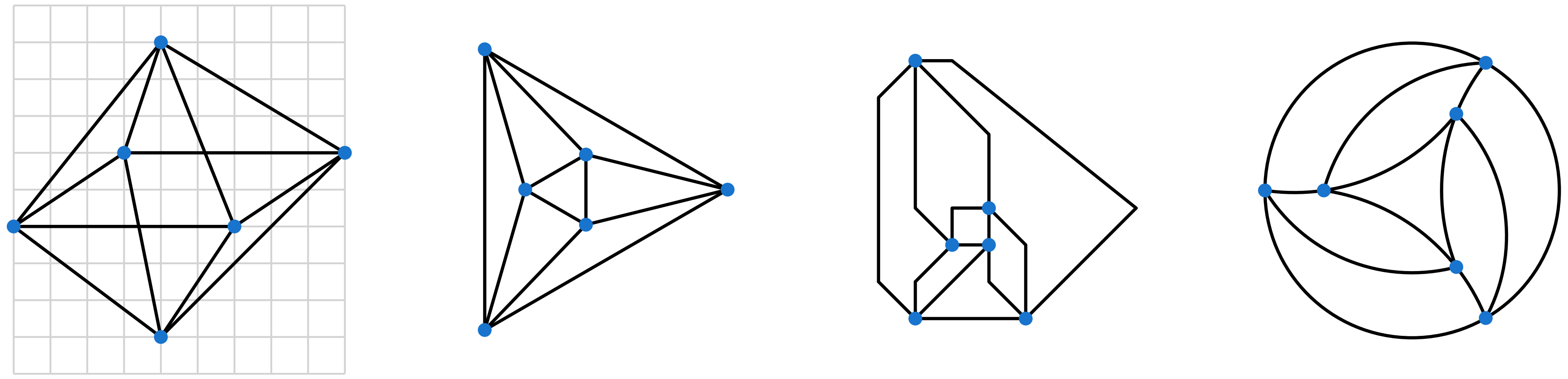
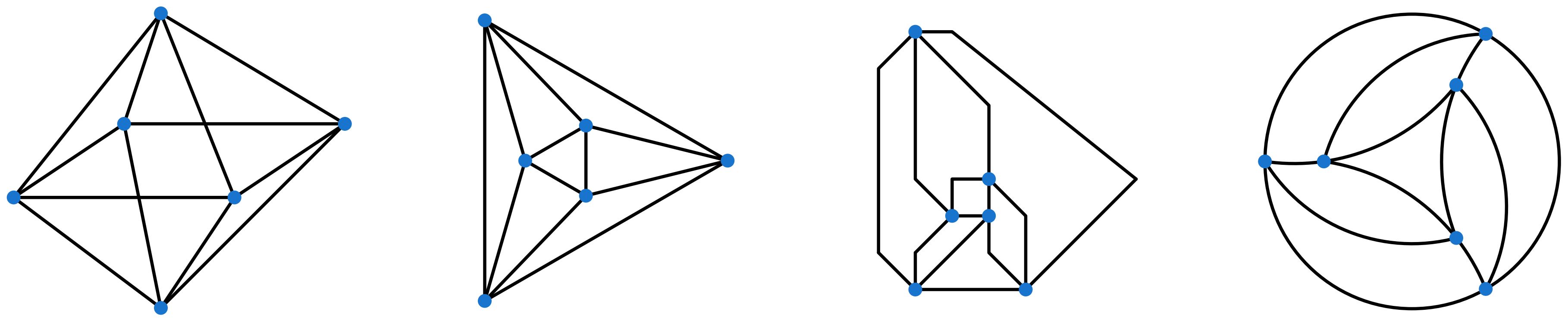


- straight-line vs. curved
- straight-line vs. polyline
- restricted slopes
- restricted to grid points

Drawing Styles



- straight-line vs. curved
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- directed drawings

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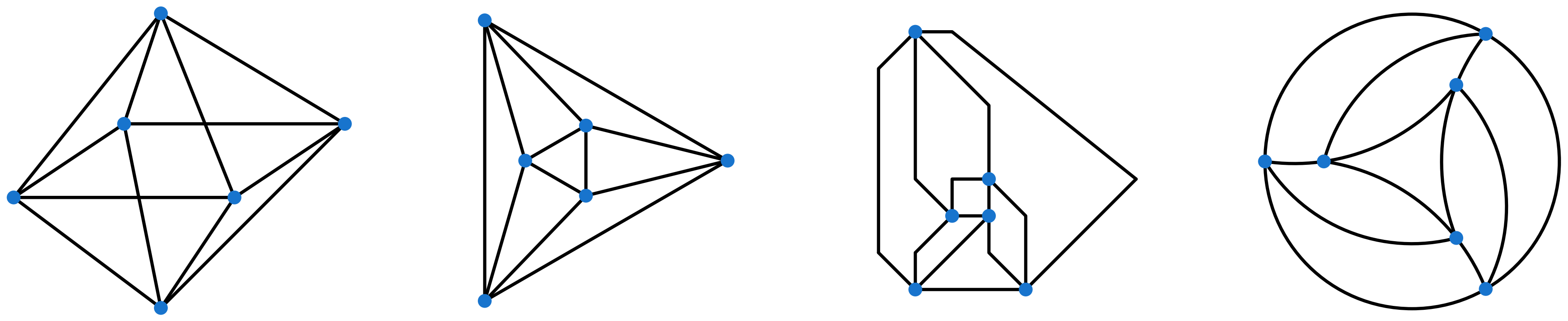


- straight-line vs. curved
- straight-line vs. polyline
- restricted slopes
- restricted to grid points
- directed drawings
- monotone drawings, confluent drawings, partial edge drawing, radial drawings, thick drawings, Lombardi drawings, ...

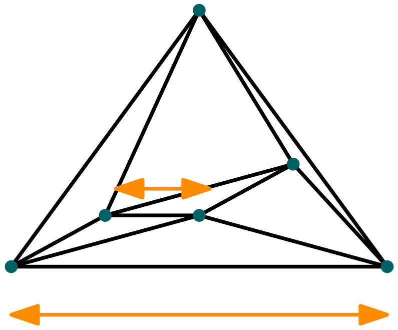
Classical Measures

Classical Measures

- vertex resolution

Classical Measures

- vertex resolution $= \frac{\text{maximal distance between two vertices}}{\text{minimal distance between two vertices}}$



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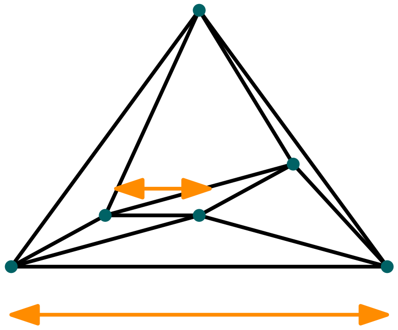

goal: small vertex resolution

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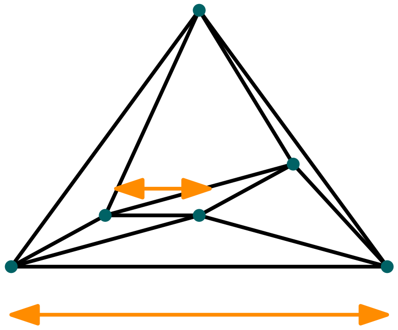

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- angular resolution

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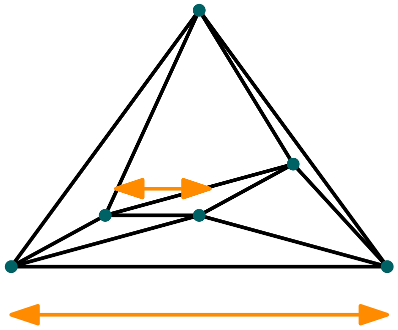

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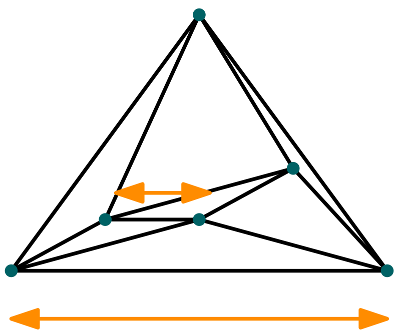

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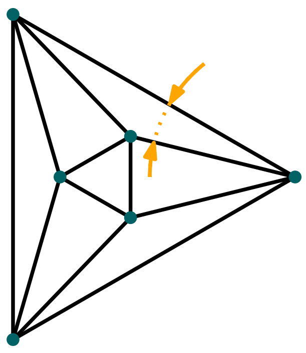

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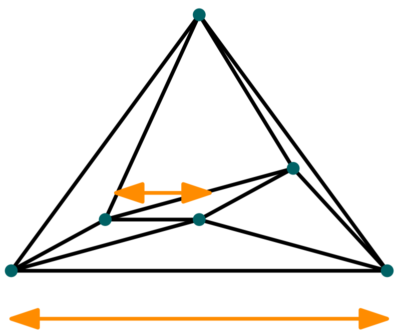

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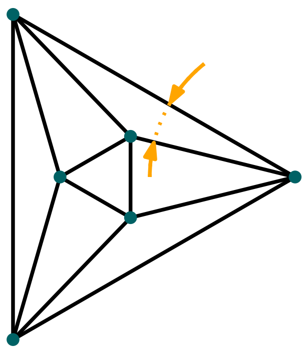

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More Measures

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- grid size

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- grid size = area of the drawing using integer grid points

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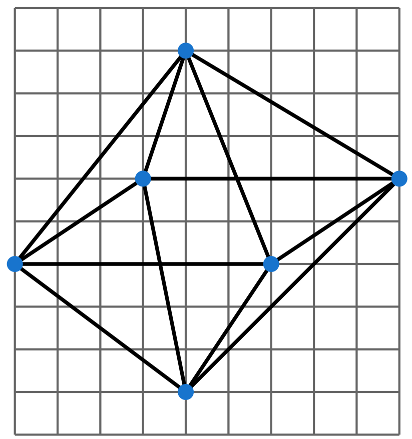

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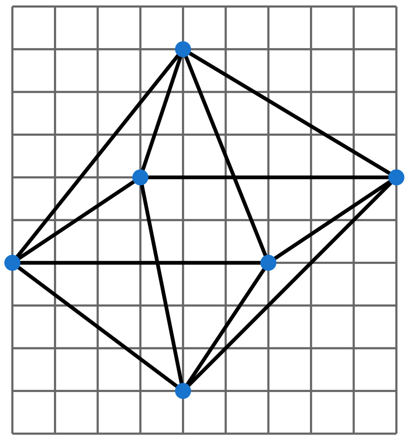

goal: small grid size

$\rightarrow$ implies good vertex and angular resolution

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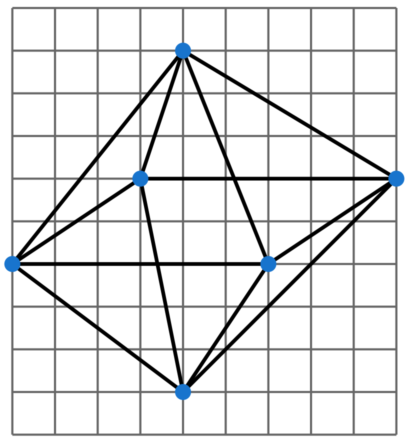

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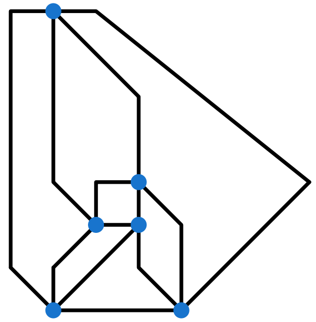
- number of bends



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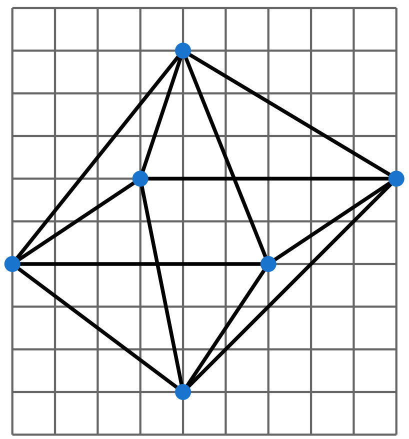

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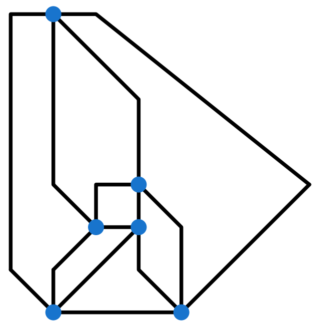

goal: minimize the number of total bends

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More Measures

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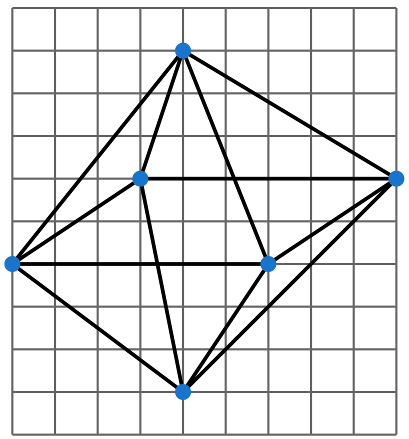

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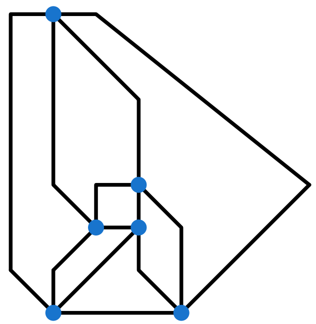

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More Measures

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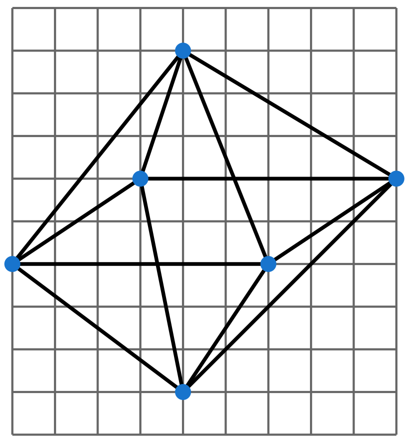

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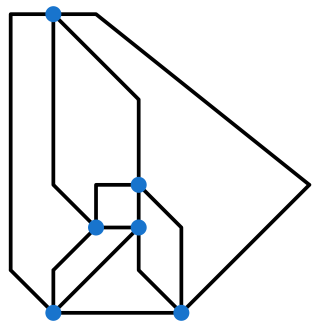

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- and many more .....

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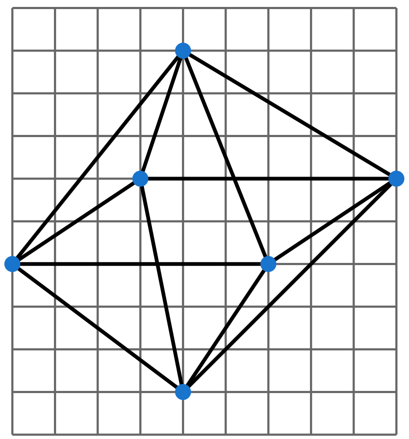

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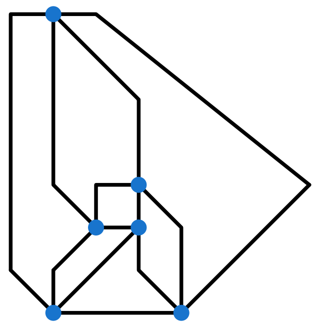

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Improving on one measure often decreases another measure!

Graph classes

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Many problems become feasible or meaningful, only when the graph class is restricted:

- trees (connected, no cycles)
- planar graphs (can be drawn without crossings)
- triangulations (maximal planar)
- planar 3-trees
- outerplanar graphs
- serial-parallel graphs
- k-connected
- ...

Prominent graph classes by example

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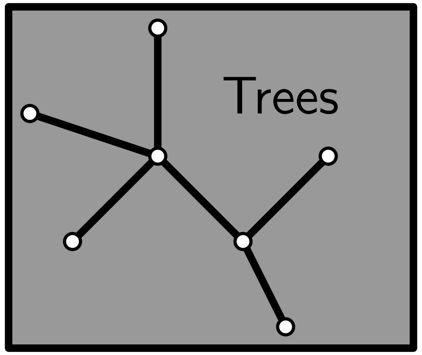

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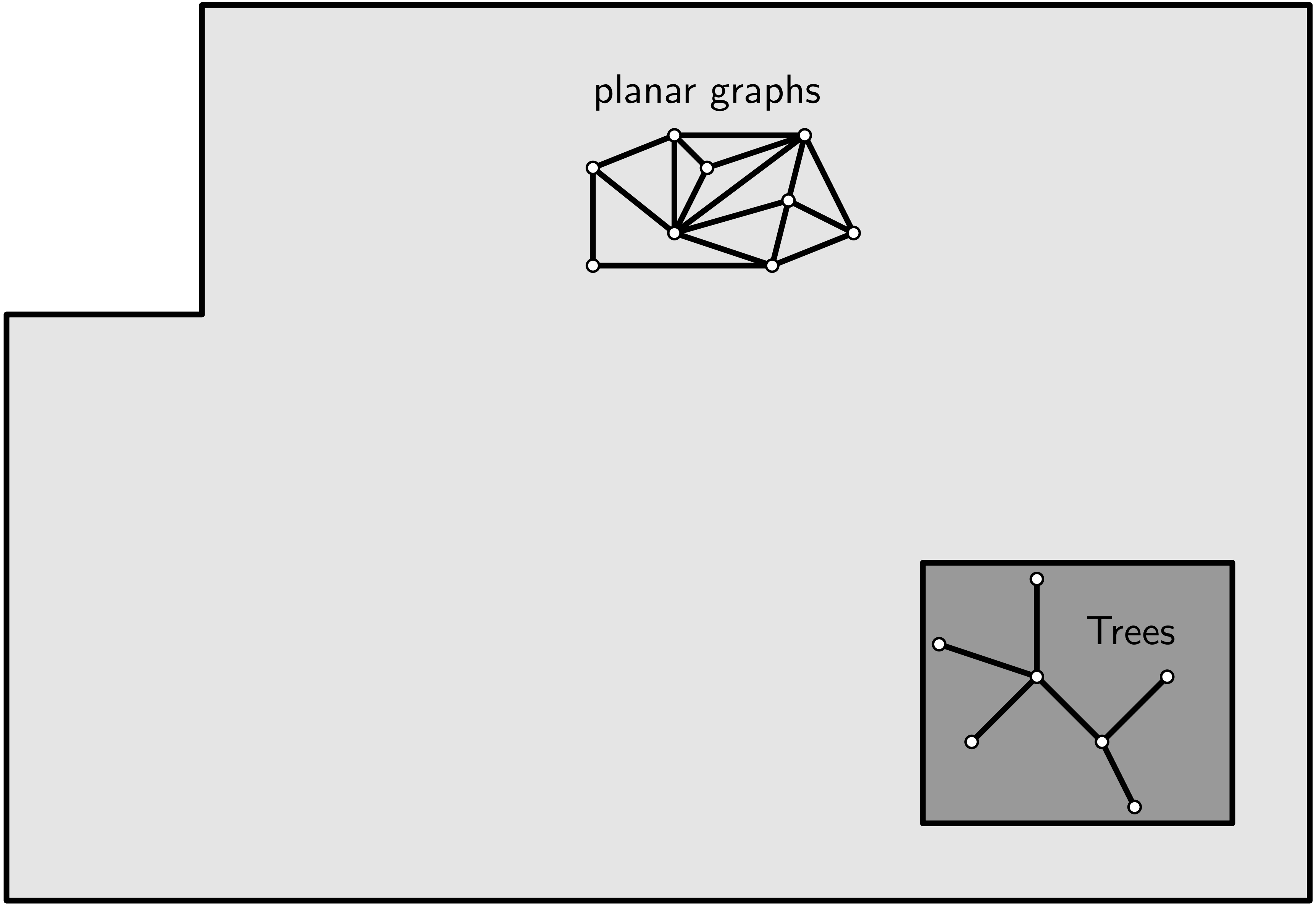

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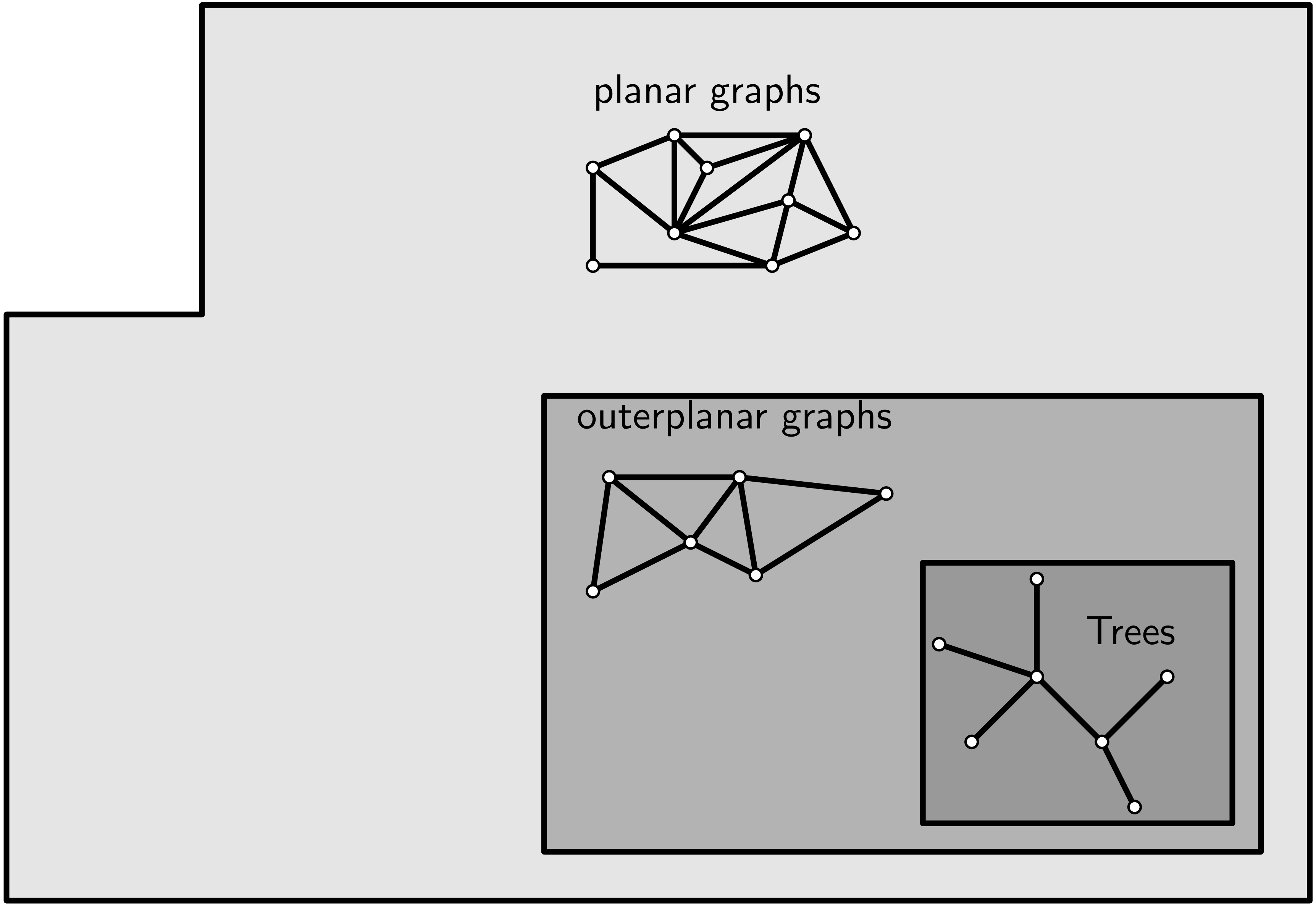

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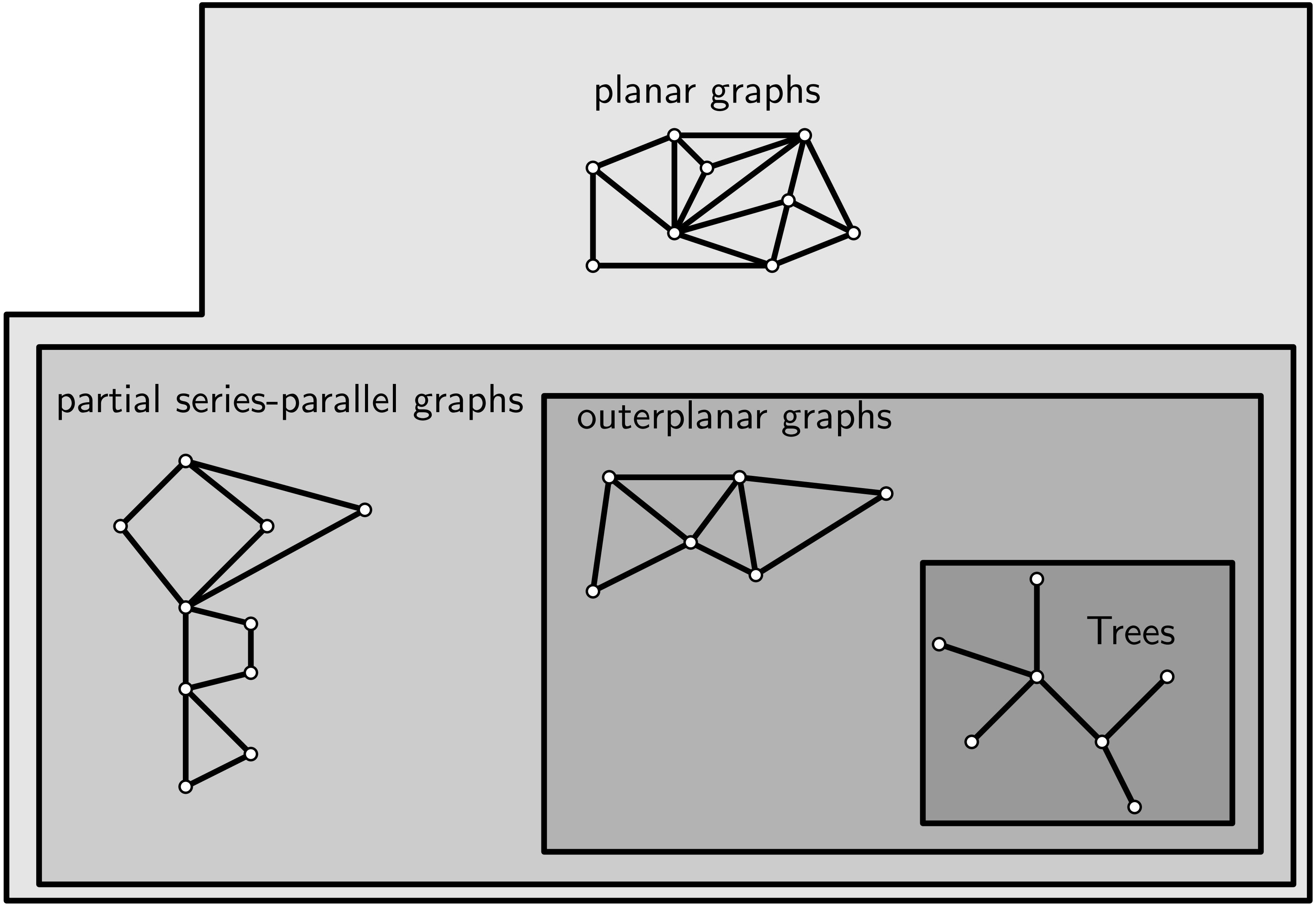

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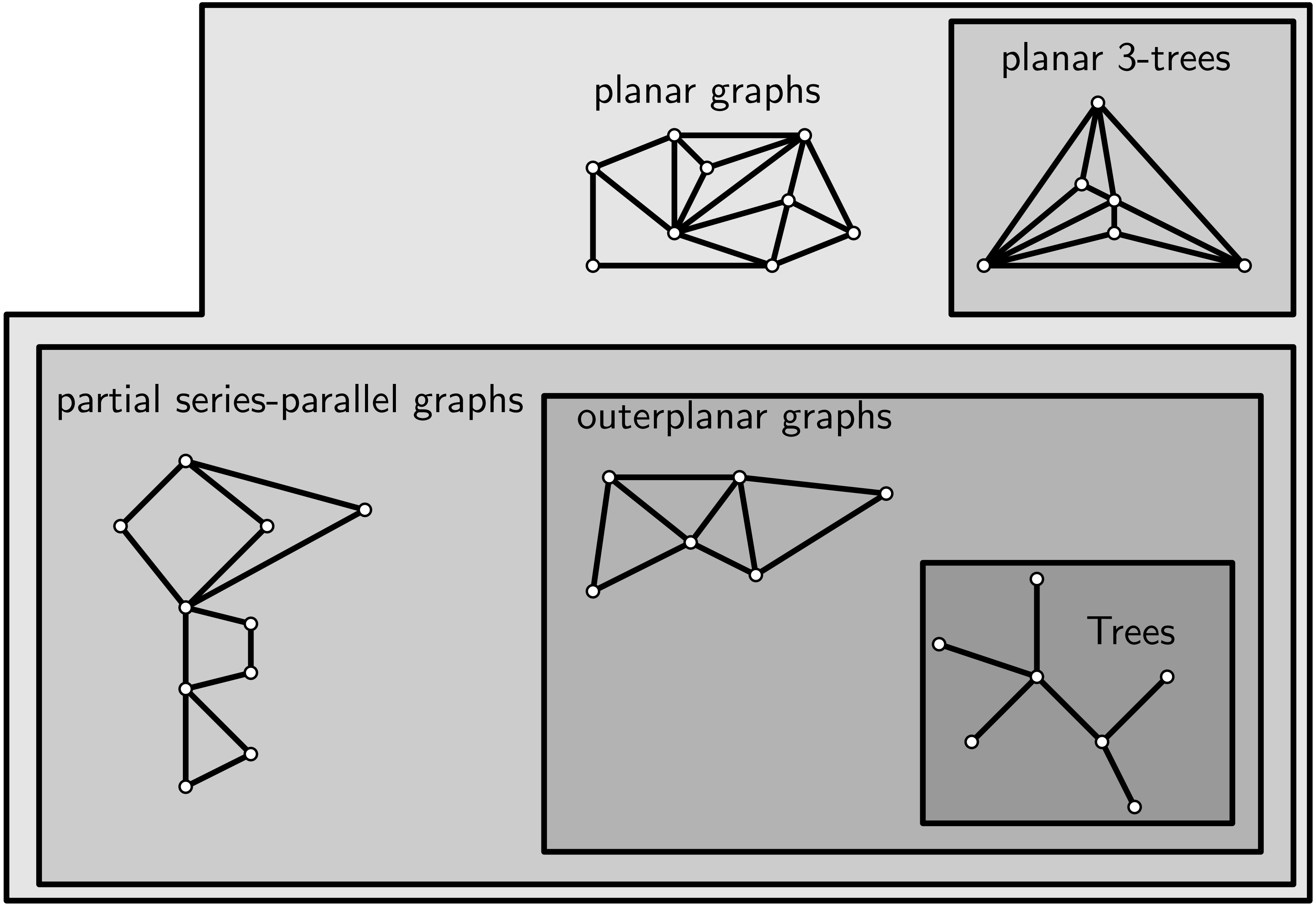

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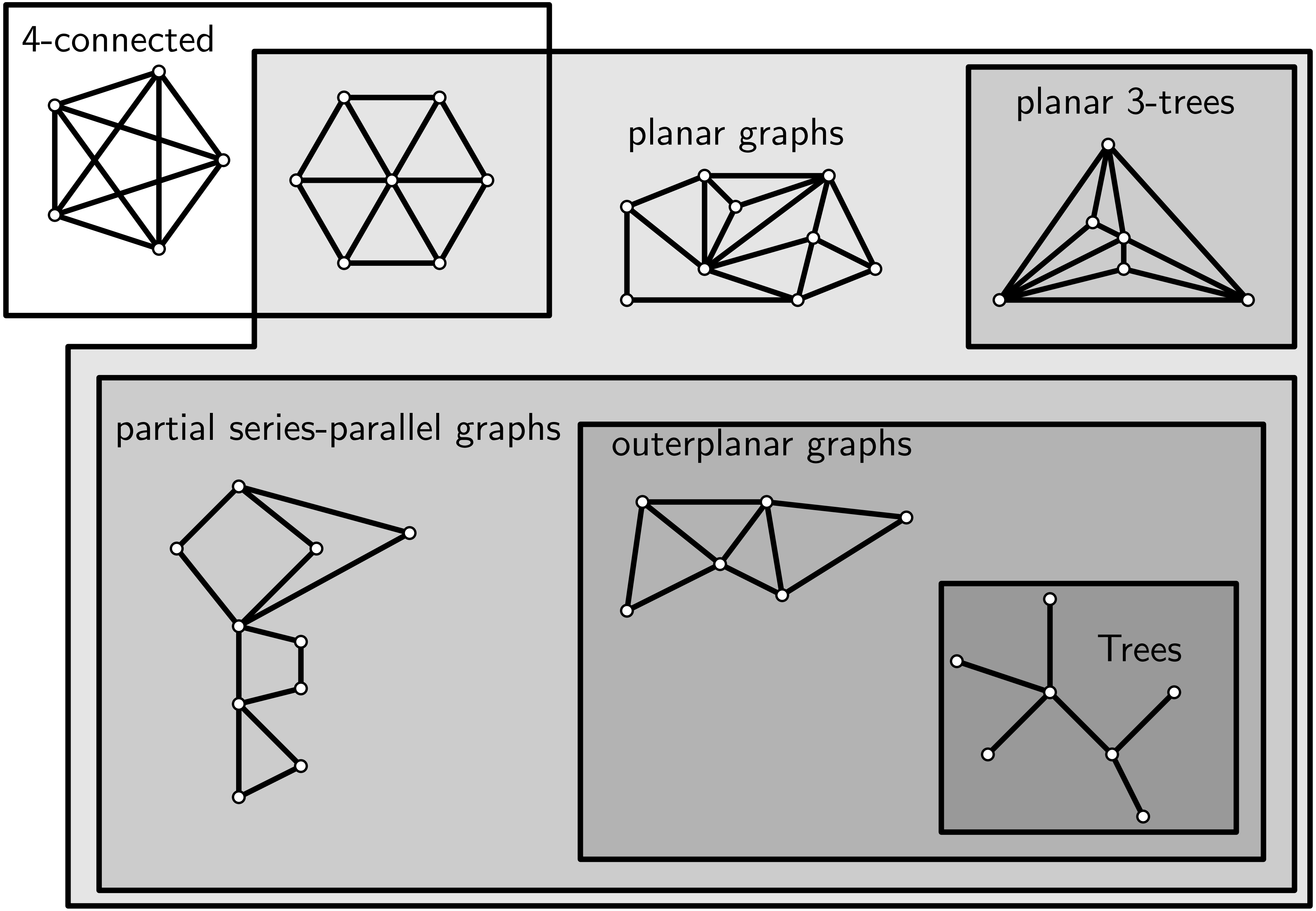

Standard techniques I

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- If your graph class has a recursive description, construct the graph drawing recursively.

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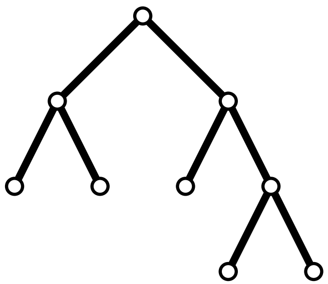
Binary trees:



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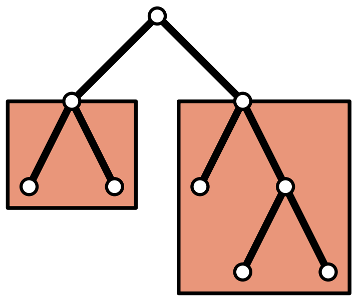

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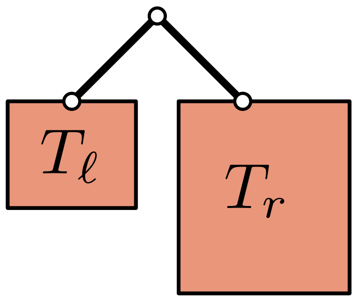

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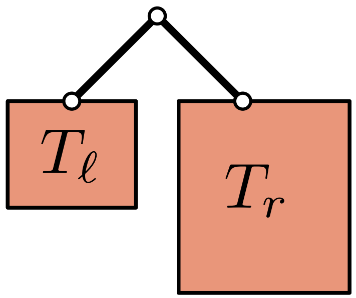

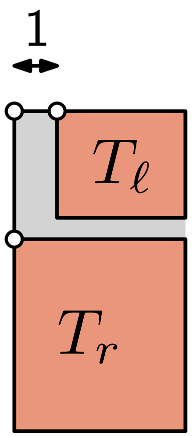

w.l.o.g. $|T_\ell| \le |T_r|$

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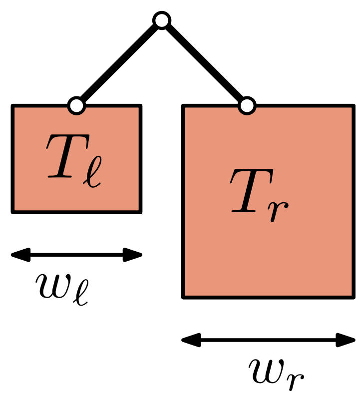

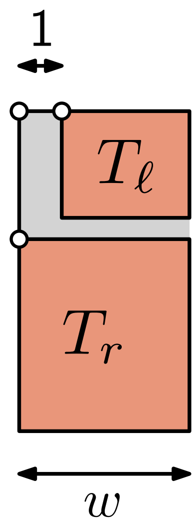

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$$w = \max\{w_l + 1, w_r\}$$

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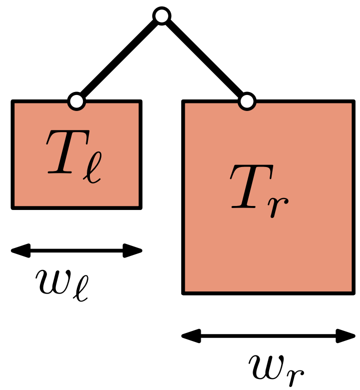

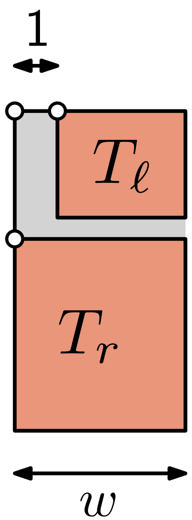

w.l.o.g. $|T_\ell| \leq |T_r|$

$w = \max\{w_l + 1, w_r\}$

$$W(n) \leq W(n/2) + 1$$
$$W(n) = O(\log n)$$

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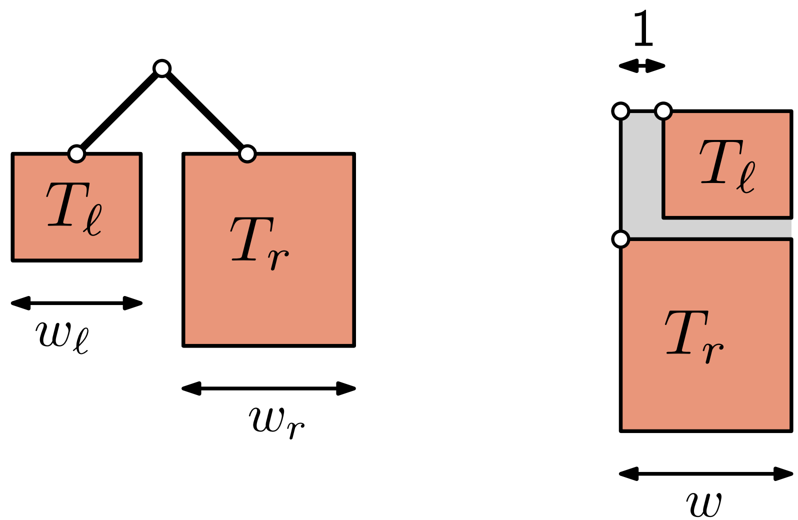

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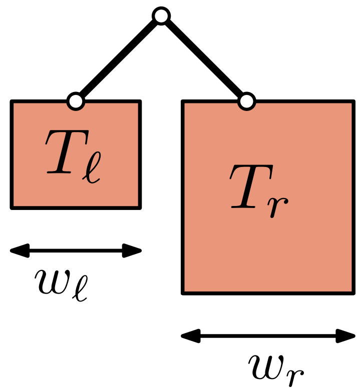

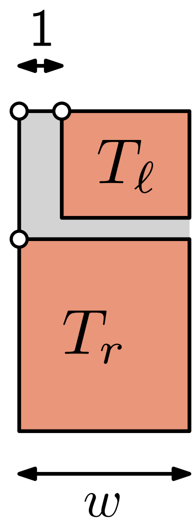

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[Crescenzi, Di Battista, Piperno '92]

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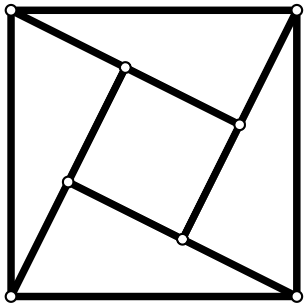

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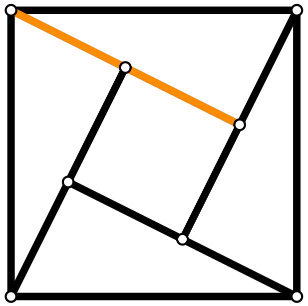

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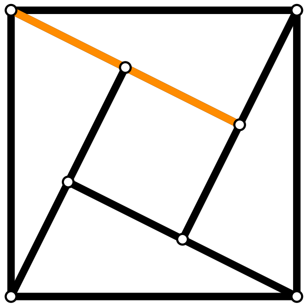

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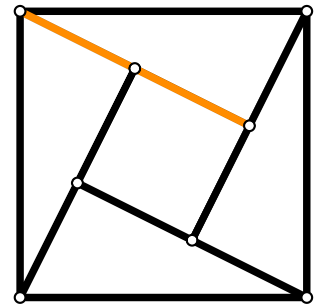

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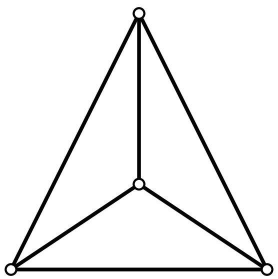

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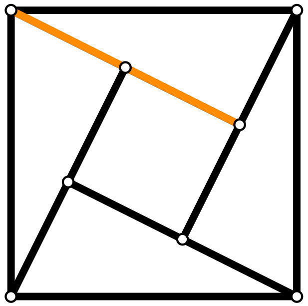

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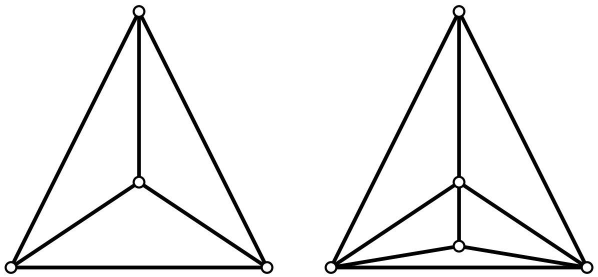

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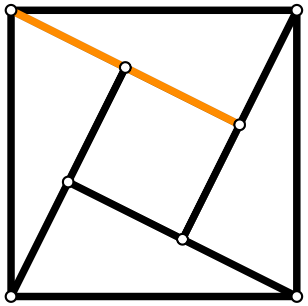

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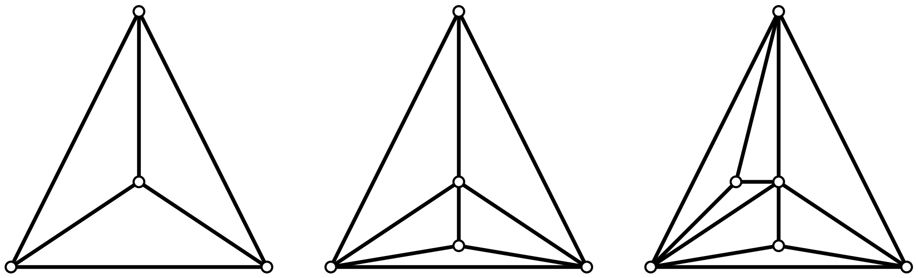

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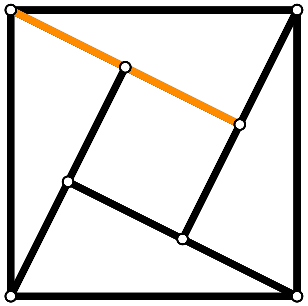

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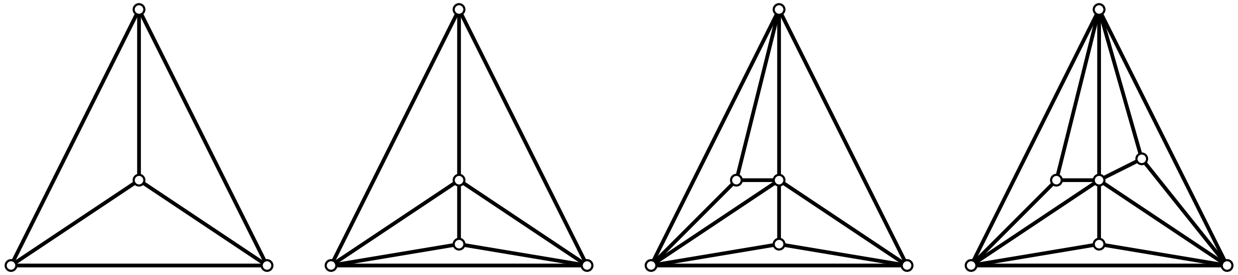

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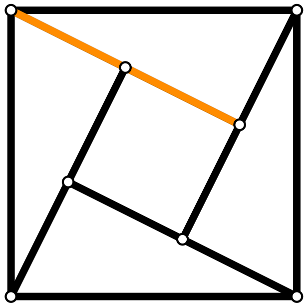

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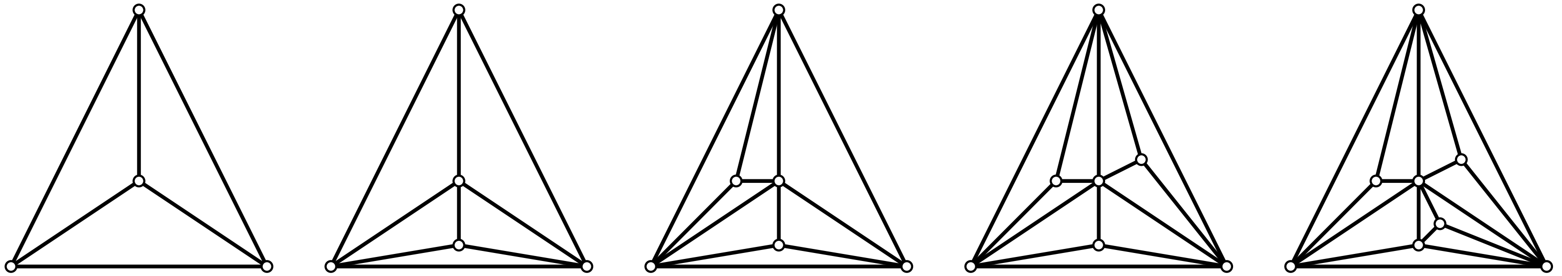

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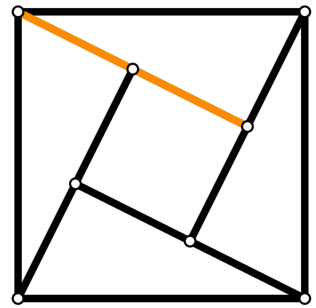

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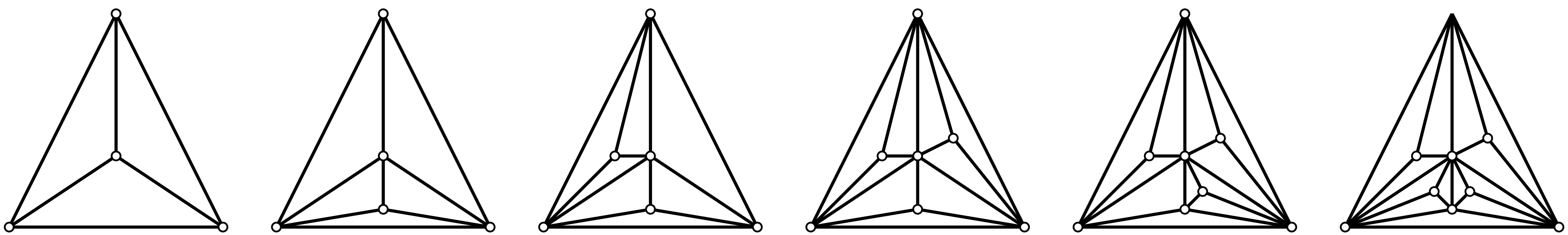

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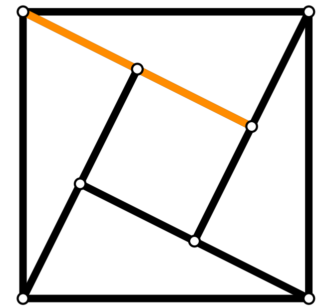

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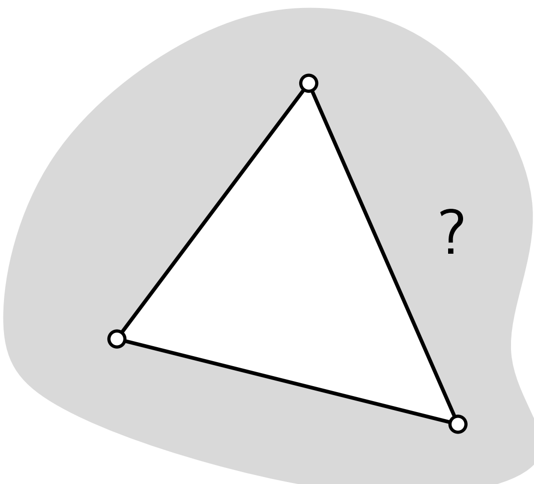

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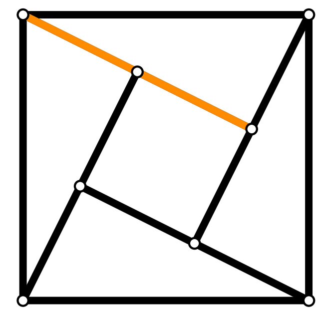

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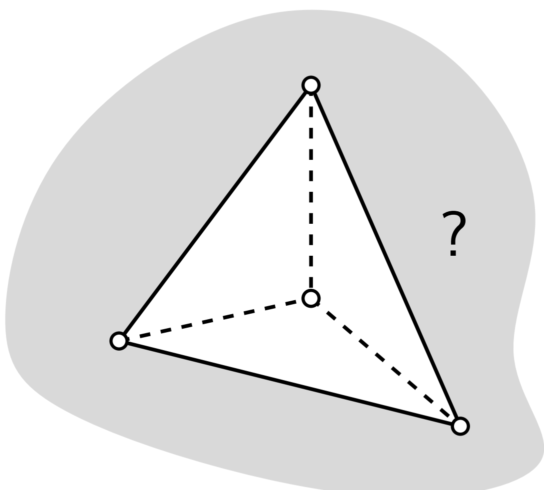

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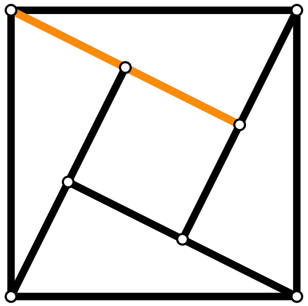

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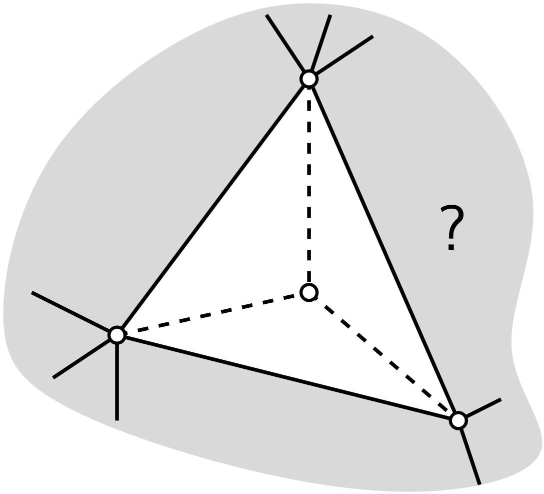

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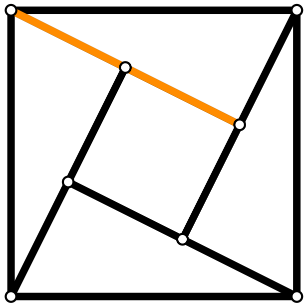

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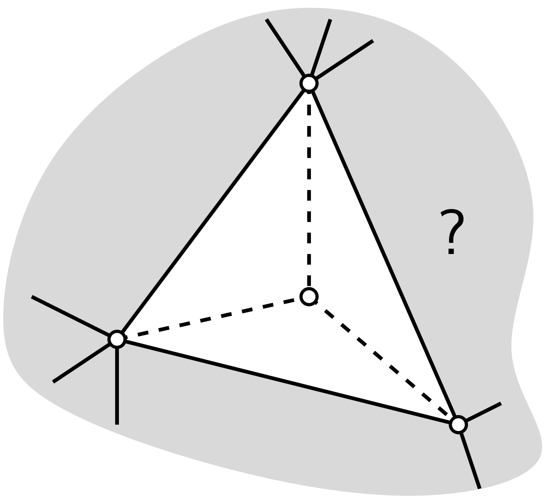

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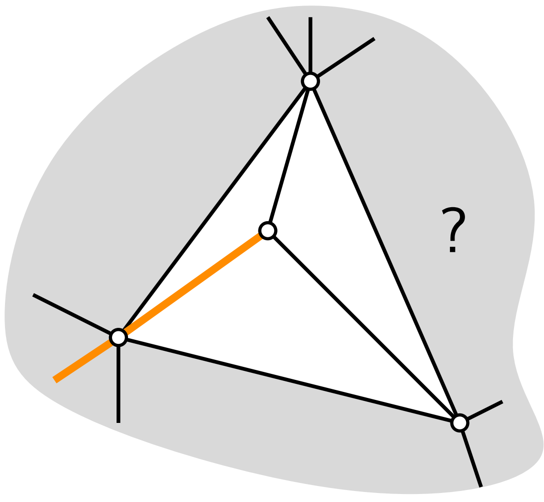

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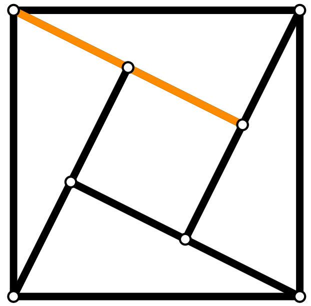

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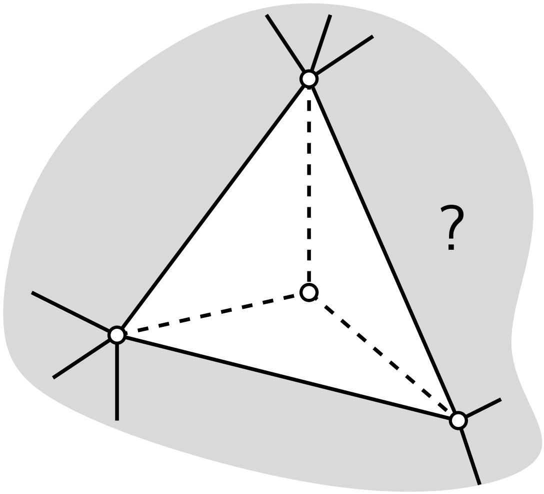

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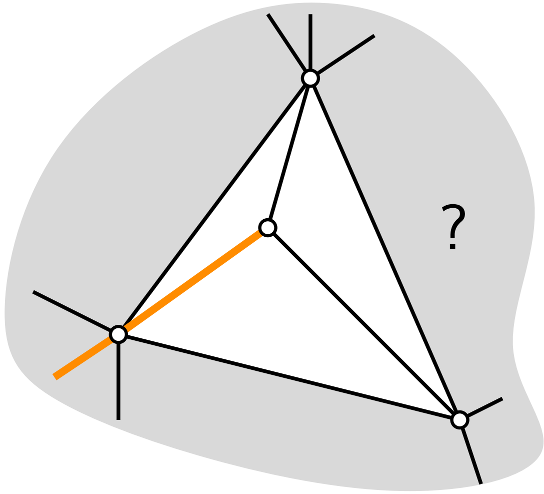

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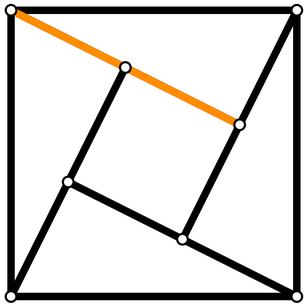

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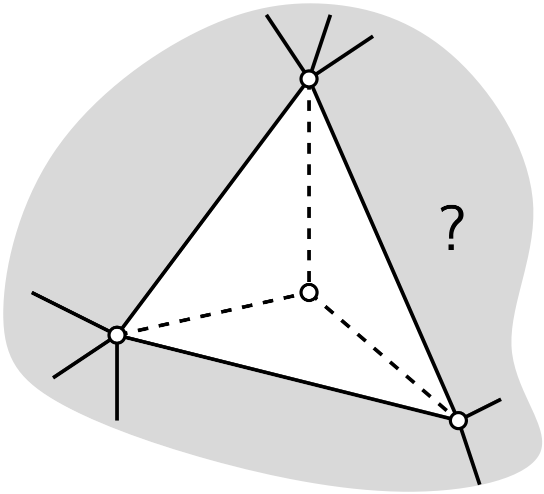

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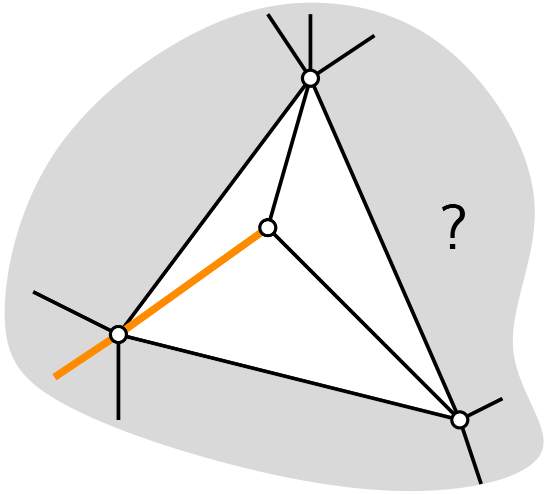

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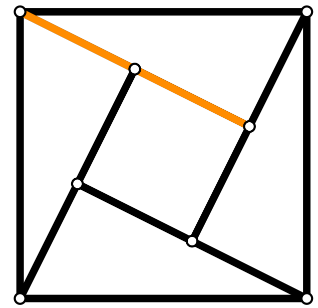

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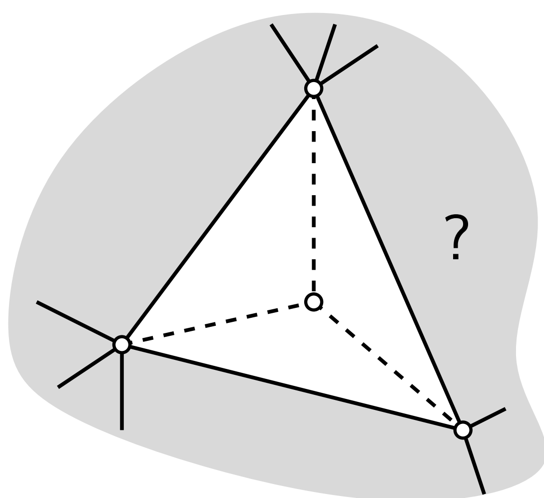

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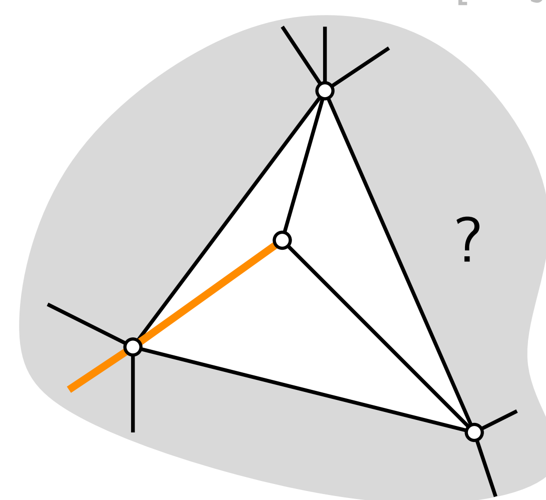

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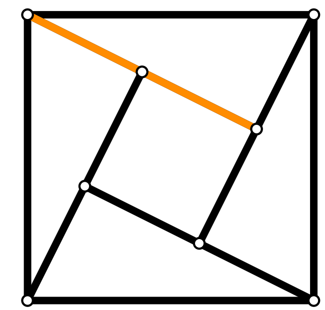

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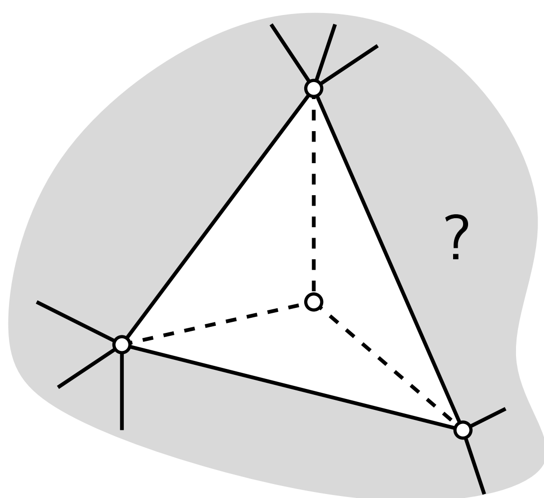

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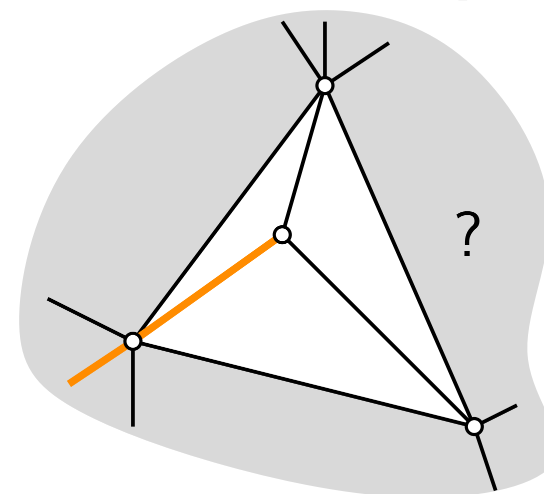

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Fruchterman Reingold '92

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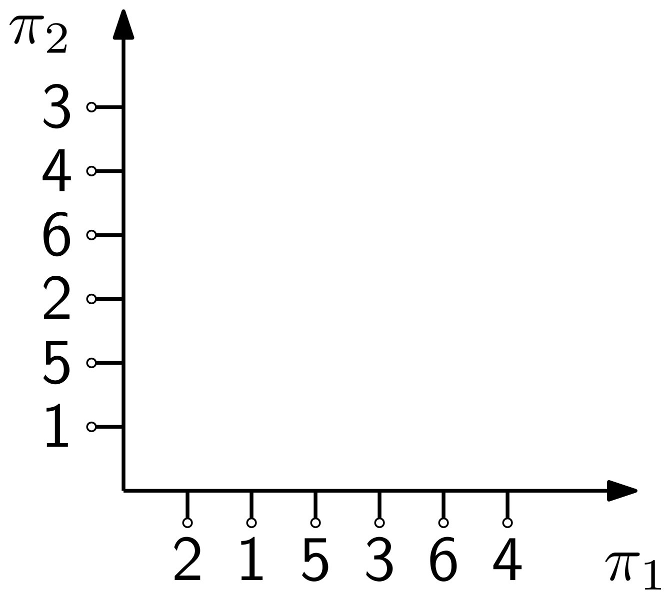

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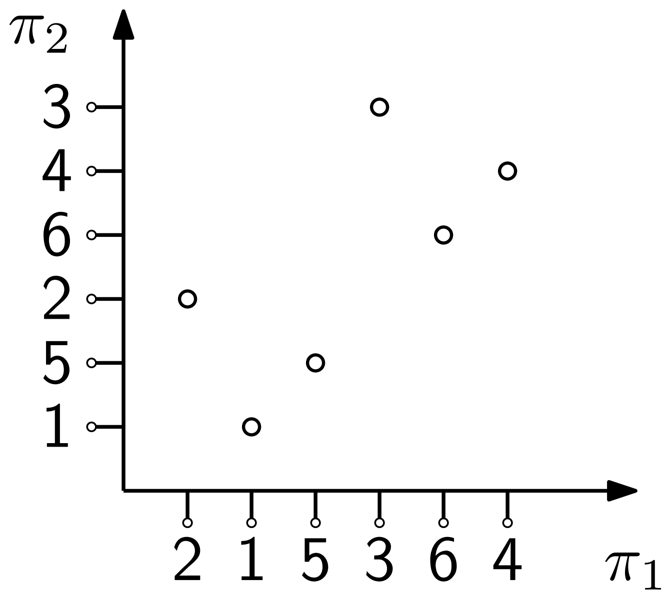

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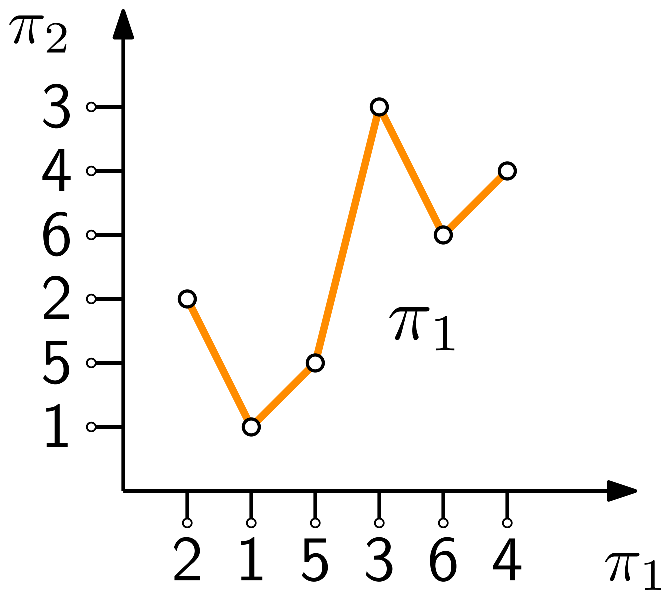

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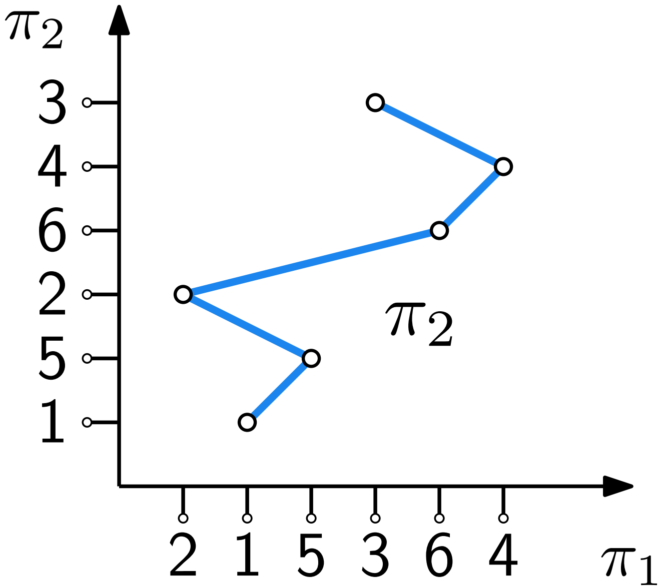

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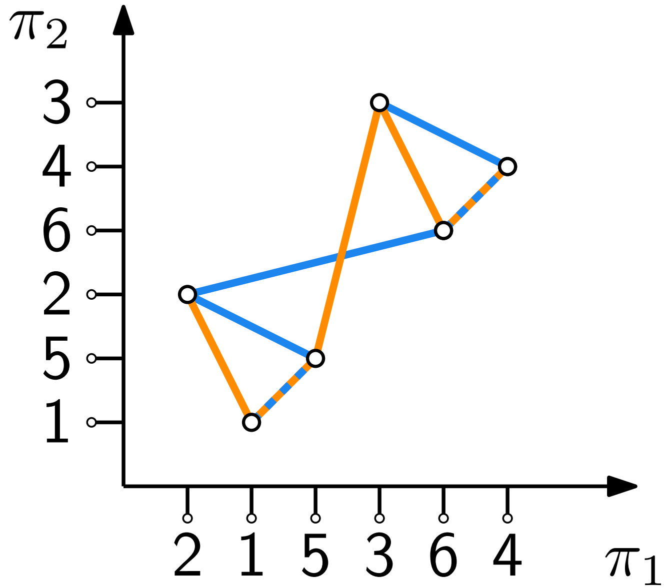

Simultaneous embedding

Given: Graphs $G_1 = (V, E_1), G_2 = (V, E_2), \ldots$.

Question: Can we place V such that each of these graphs has a planar (straight-line) drawing?

Two paths: YES

[Brass et al. '07]

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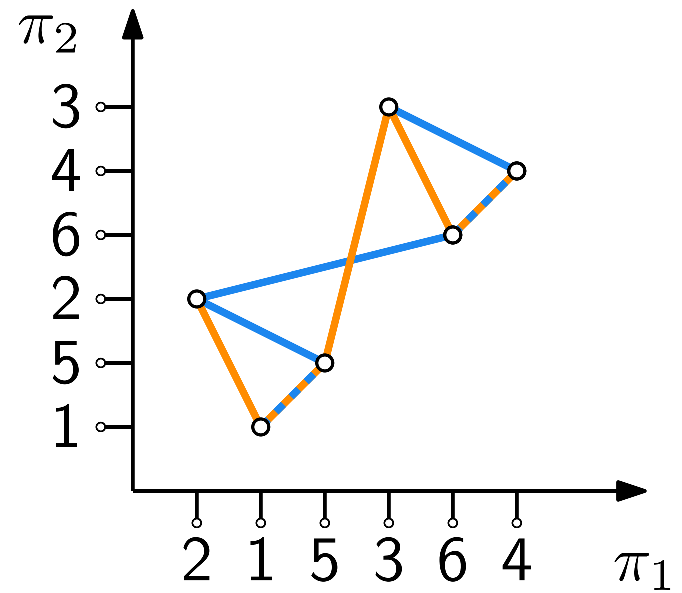

Six Matchings: NO

[Cabello et al. '07]

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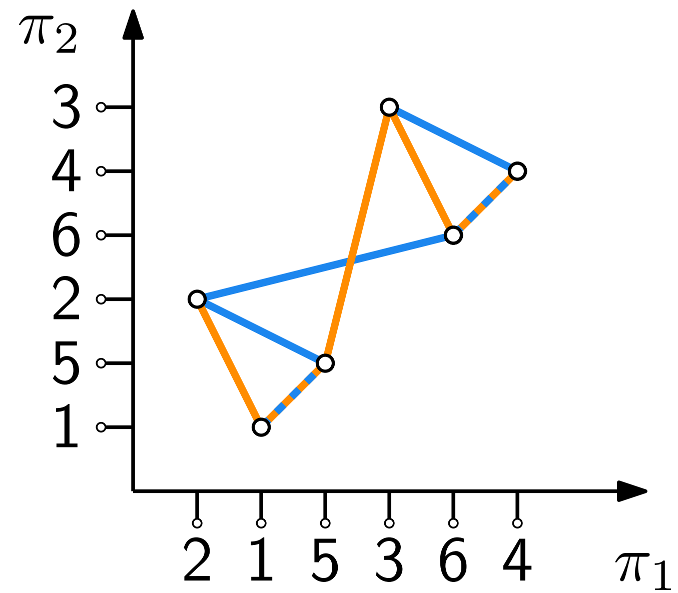

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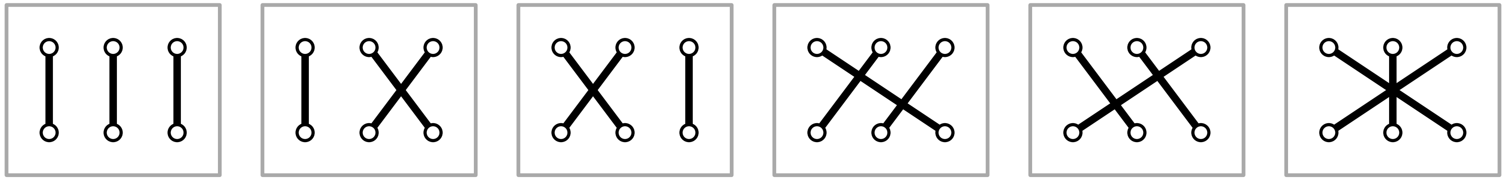

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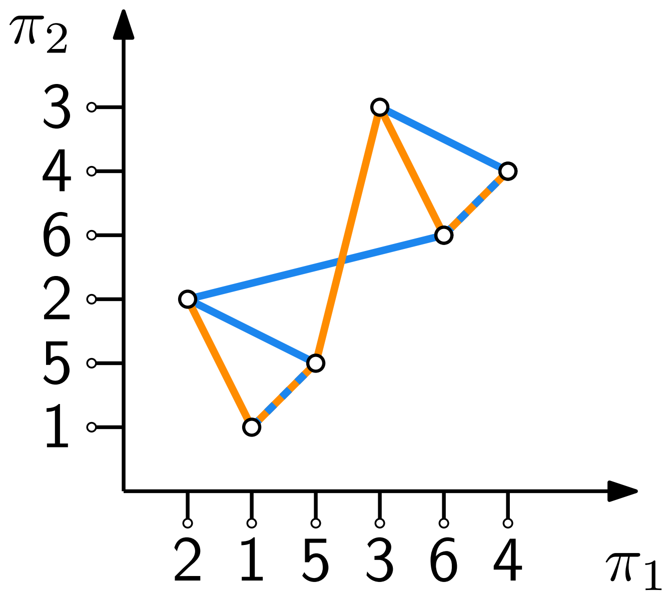

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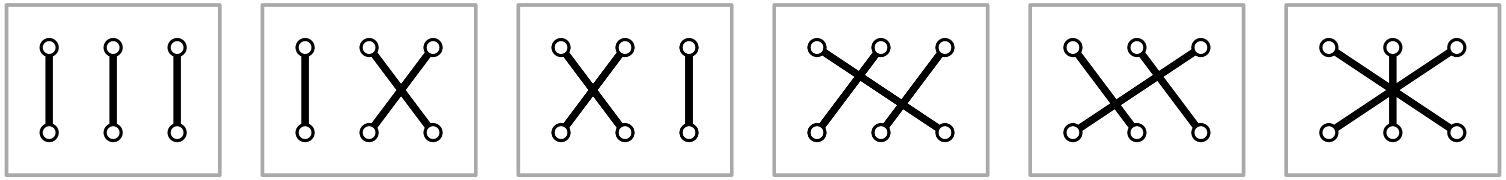

In a $K_{3,3}$-drawing at least two edges cross. For every pair of edges one matching contains these.

Morphing

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Not ONE (geometric) graph with vertices in different positions?

Given: Drawings of planar graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$.

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- Compute (asymmetric) spring weights for the drawings of G_1/G_2.
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Alternative Solution: [Angelini et al. '14]:

- linear number of linear moves per vertex (worst-case opt.)
- complicated

3D

Given a graph G, find a set of planes in 3-space such that there is a *crossing-free straight-line drawing* of G with all vertices and edges drawn on these planes.

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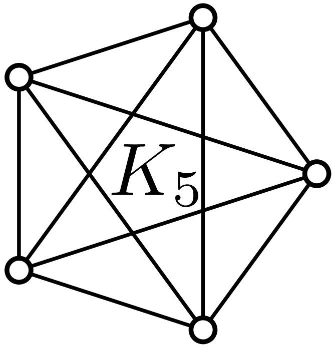

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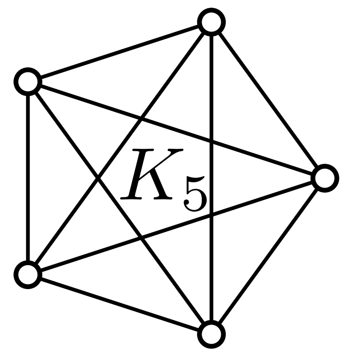

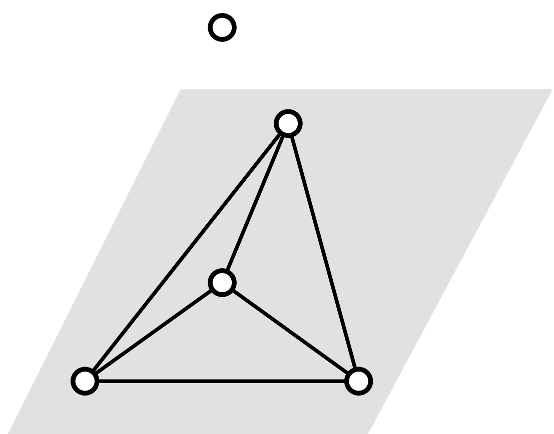

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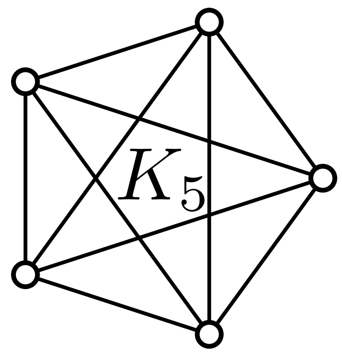

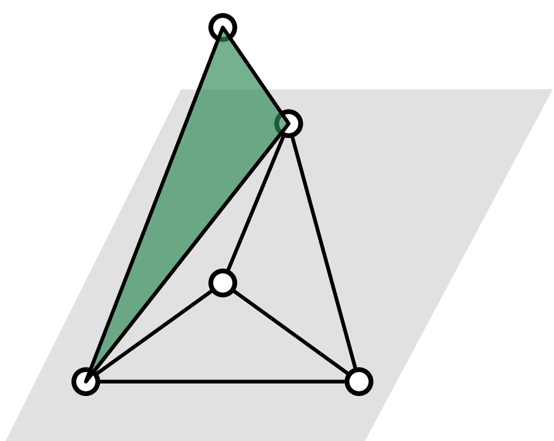

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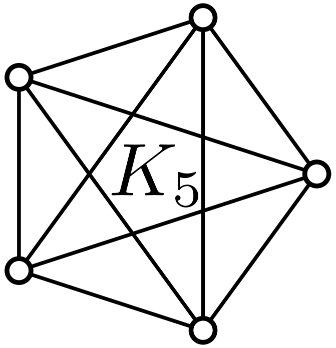

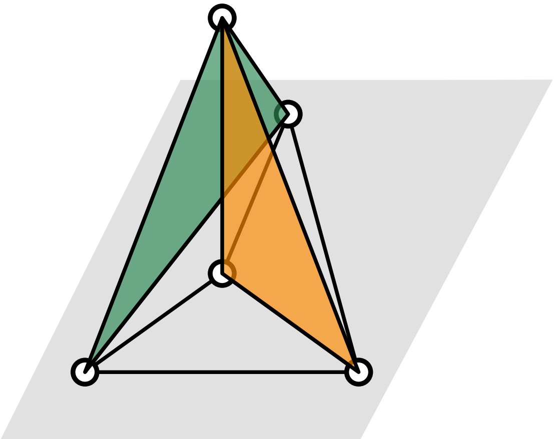

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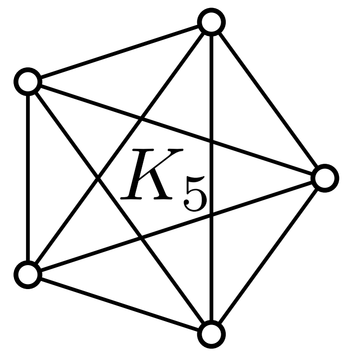
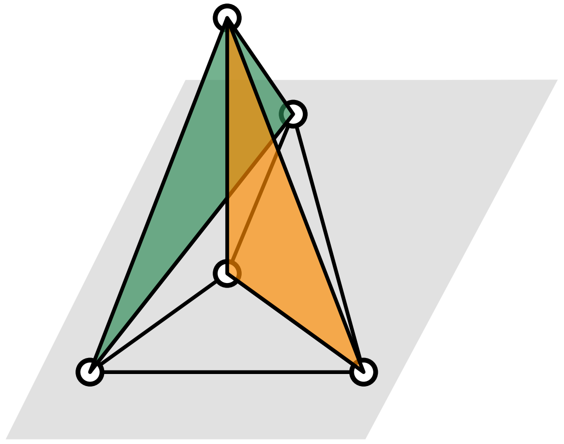
$$\rho_3^2(K_5) = 3$$





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$$\rho_3^2(K_6) = ?$$

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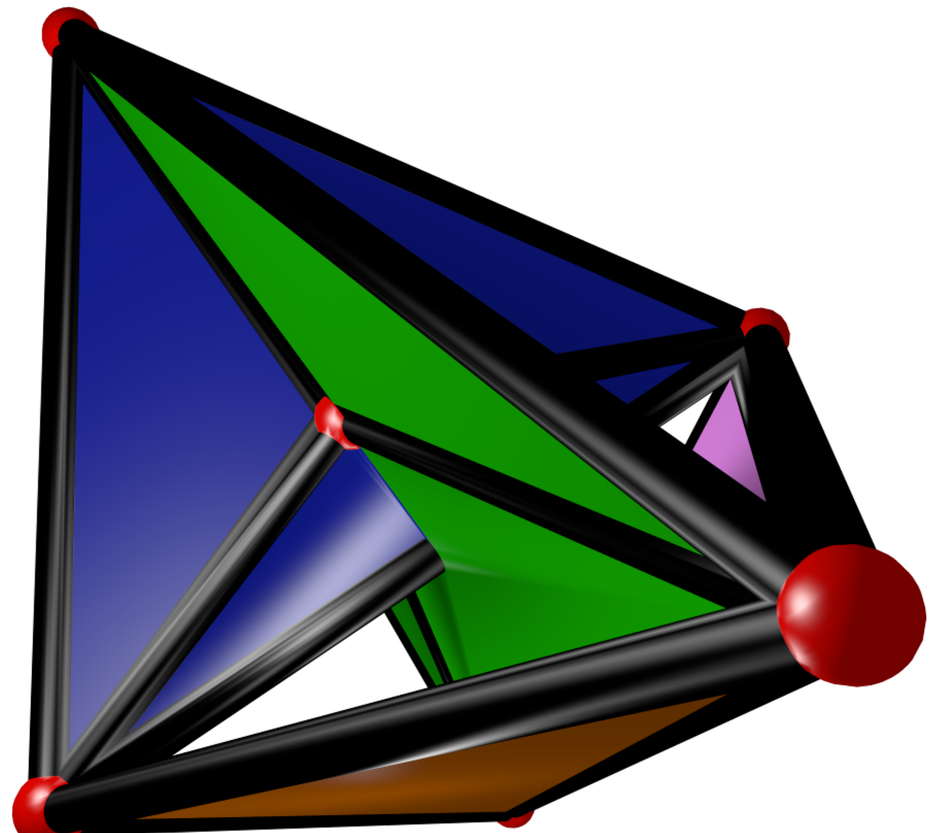

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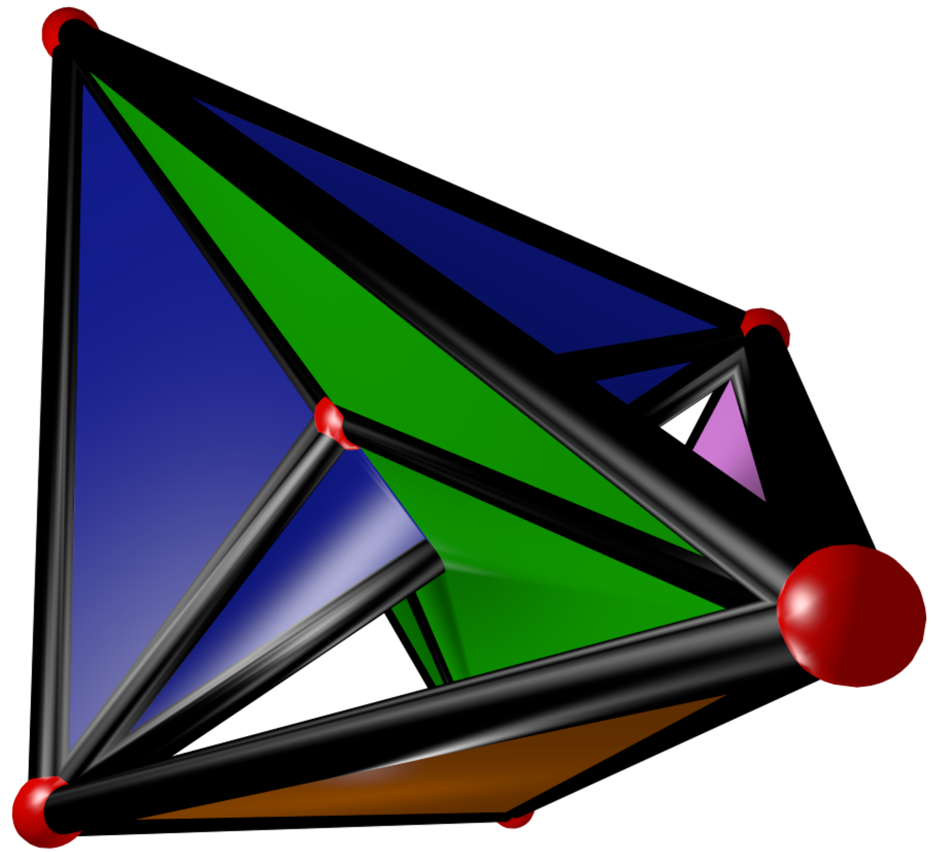

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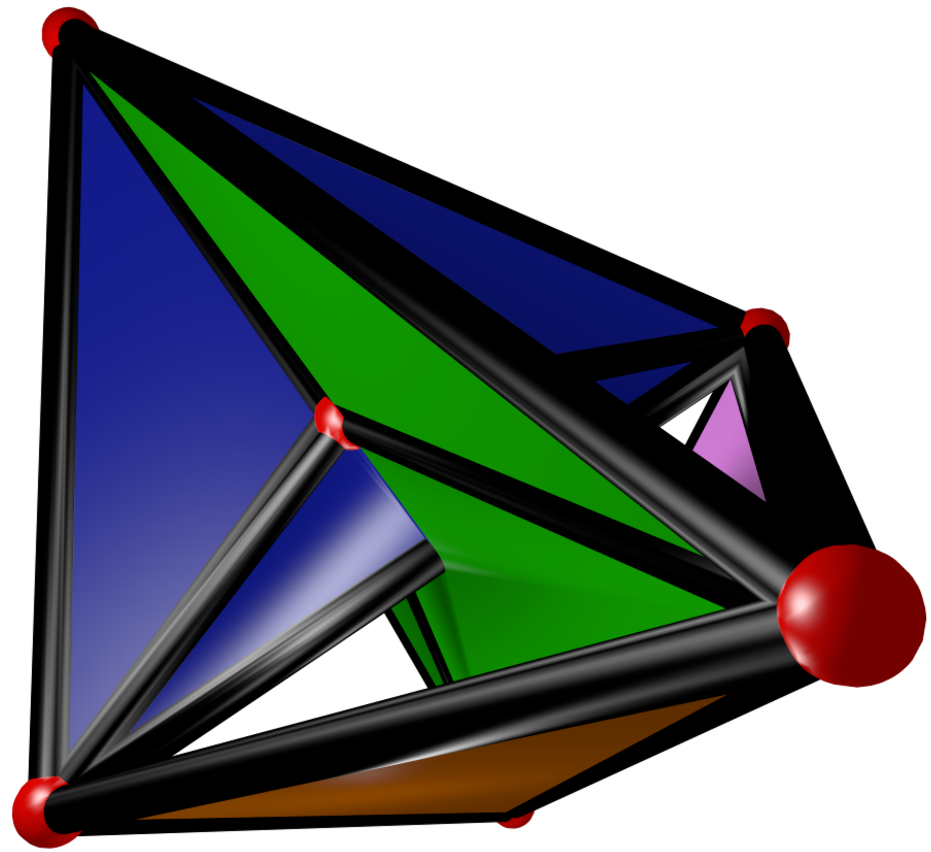

For any planar graph G, clearly $\rho_3^2(G) = 1$.

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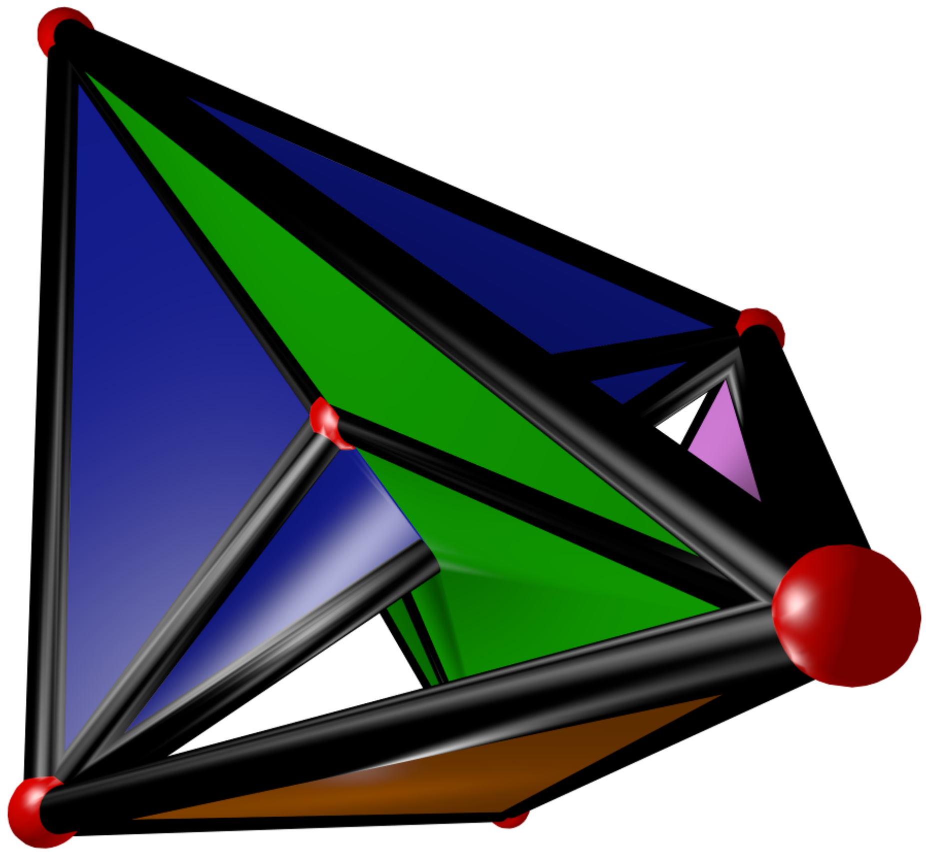

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Note: $\rho_3^2(K_n) \in \Theta(n^2)$.

$$\binom{n}{2}/6 \lessapprox \rho_3^2(K_n) \lessapprox \binom{n}{2}/3$$

Affine Cover Numbers

Let G be a graph and $1 \leq m < d$.

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minimum number of m-dimensional hyperplanes in $\mathbb{R}^d$ s.t. G has a crossing-free straight-line drawing that is contained in these planes.

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$\rho_d^m = \pi_d^m = 1$ for $m \geq 3$

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Interesting cases

- Line cover numbers in 2D and 3D: ρ_2^1, ρ_3^1, π_2^1, π_3^1

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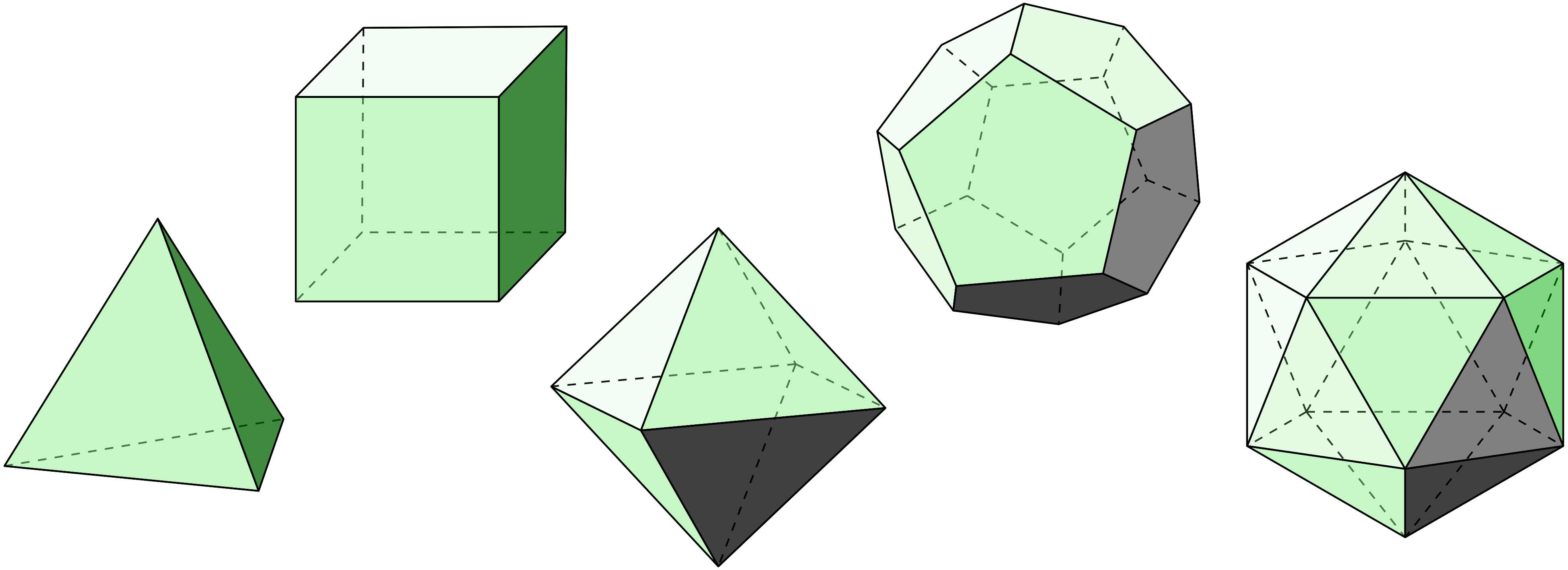
[WADS'17 & GD'19]

Unfortunately, *each* of these numbers is NP-hard to compute :-(

Line Cover Numbers of the Platonic Solids

$G = (V, E)$	$\lvert V \rvert$	$\lvert E \rvert$	$\lvert F \rvert$	$\rho_2^1(G)$	$\rho_3^1(G)$	$\pi_2^1(G)$	$\pi_3^1(G)$
tetrahedron	4	6	4				
cube	8	12	6				
octahedron	6	12	8				
dodecahedron	20	30	12				
icosahedron	12	30	20				



Line Cover Numbers of the Platonic Solids

[Kryven et al., CALDAM'18]

$G = (V, E)$	$\|V\|$	$\|E\|$	$\|F\|$	$\rho^1_2(G)$	$\rho^1_3(G)$	$\pi^1_2(G)$	$\pi^1_3(G)$
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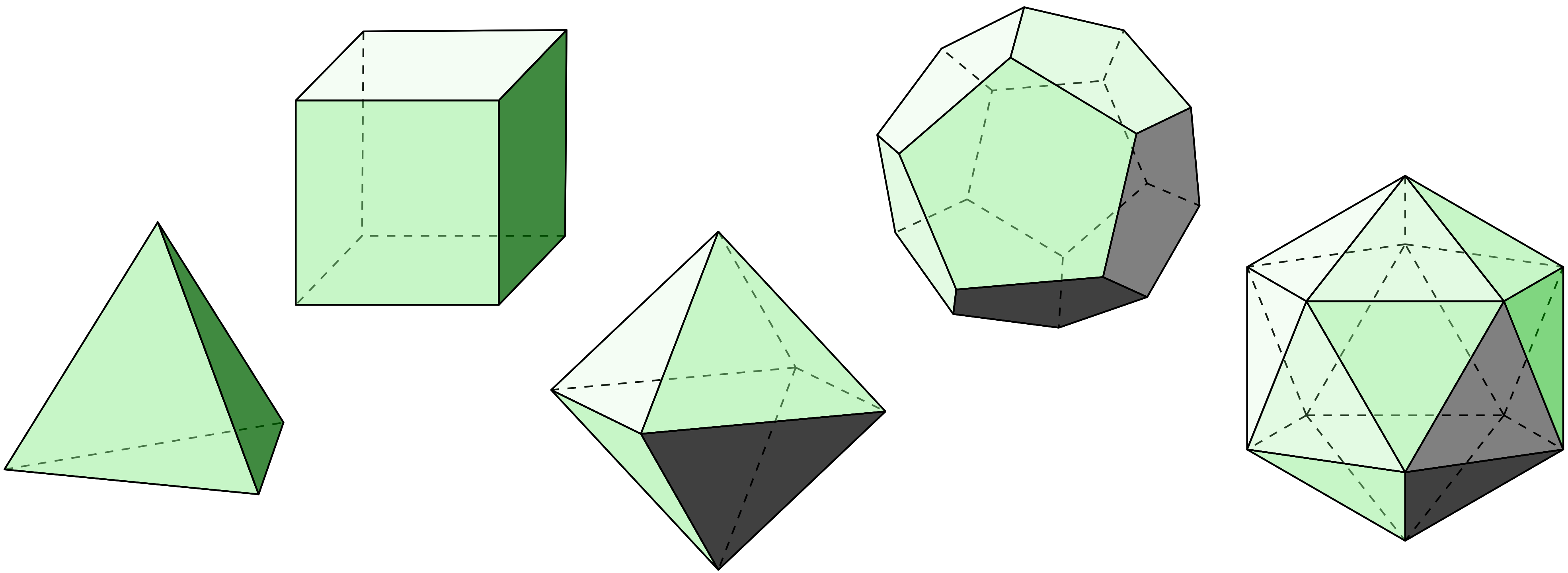

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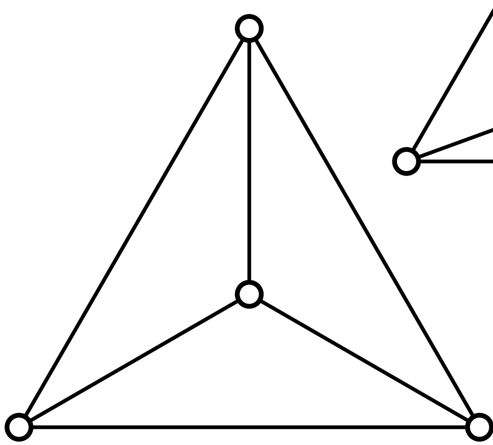
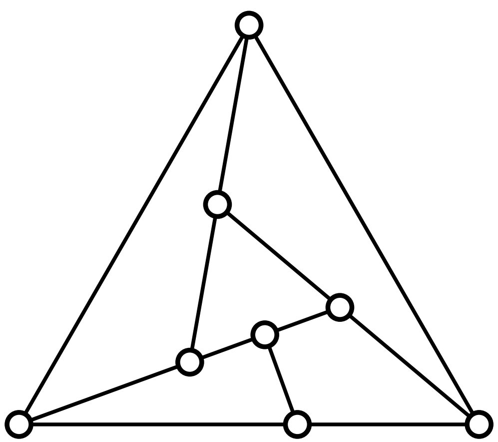
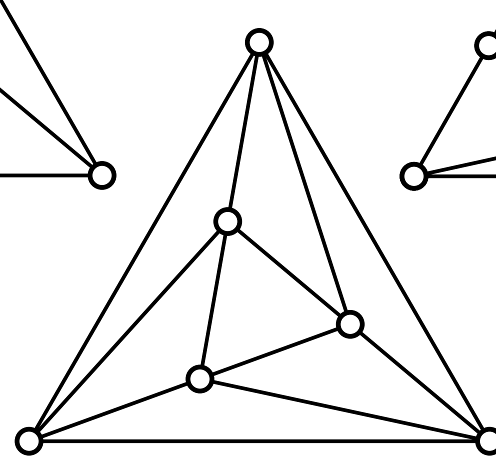
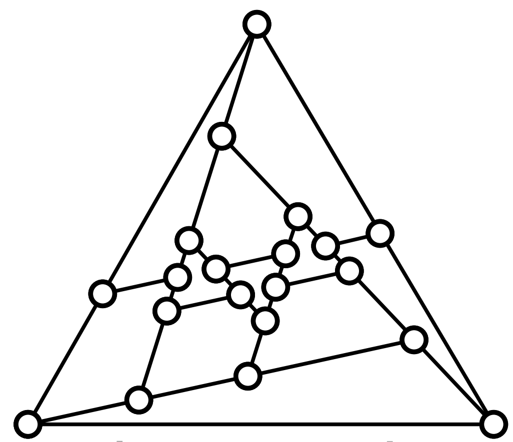
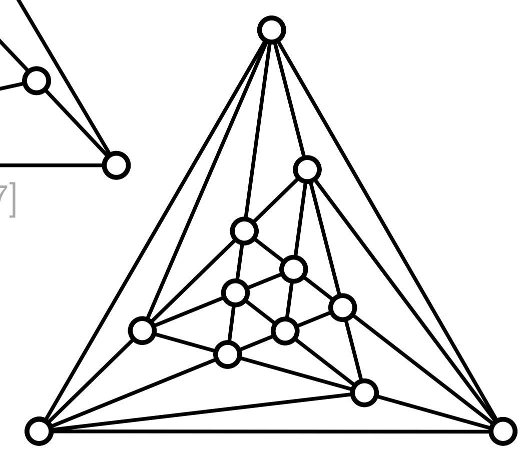
cube	8	12	6	7	7		
octahedron	6	12	8	9	9		
dodecahedron	20	30	12	9...10	9...10		
icosahedron	12	30	20	13...15	13...15		

[Scherm, B.Th.'17]



Line Cover Numbers of the Platonic Solids

$G=(V,E)$	$\lvert V\rvert$	$\lvert E\rvert$	$\lvert F\rvert$	$\rho_2^1(G)$ [Kryven et al., CALDAM'18]	$\rho_3^1(G)$ [Kryven et al., CALDAM'18]	$\pi_2^1(G)$ [Firman, MTh. '17]	$\pi_3^1(G)$ [Firman, MTh. '17]
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octahedron	6	12	8	9	9		
dodecahedron	20	30	12	9...10	9...10		
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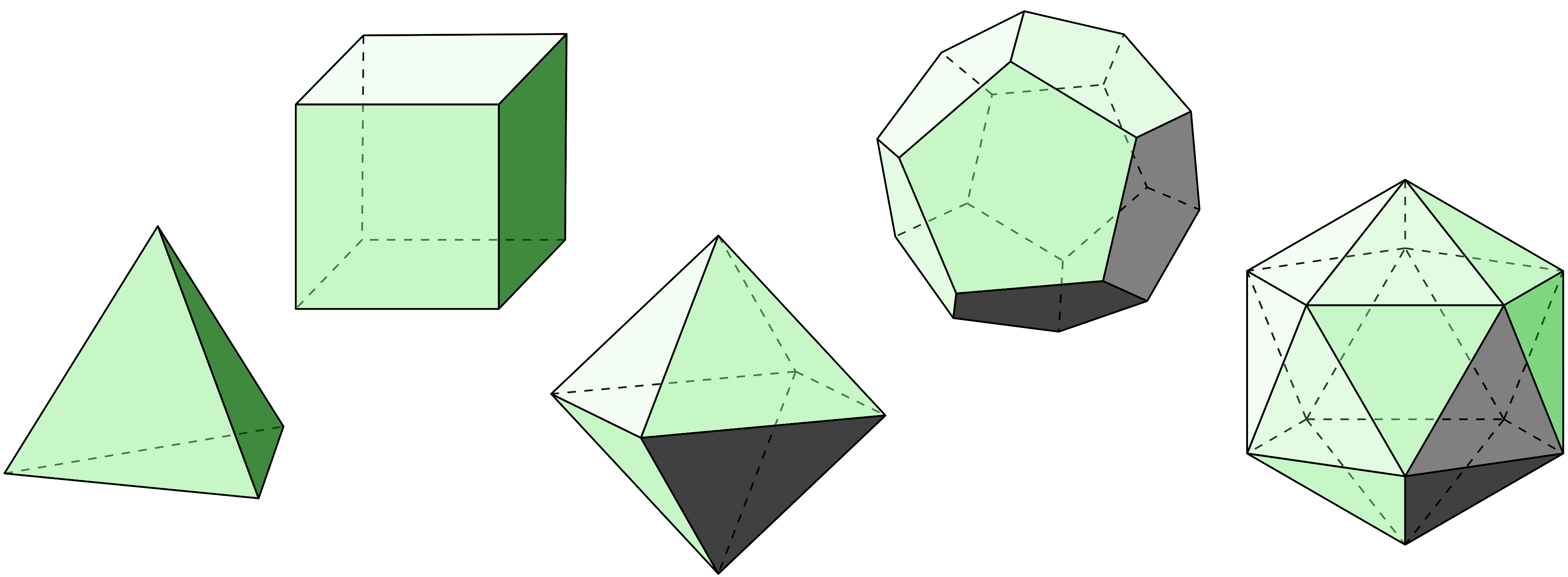

Line Cover Numbers of the Platonic Solids

[Kryven et al., CALDAM'18] [Firman, MTh. '17]

$G = (V, E)$	$\|V\|$	$\|E\|$	$\|F\|$	$\rho_2^1(G)$	$\rho_3^1(G)$	$\pi_2^1(G)$	$\pi_3^1(G)$
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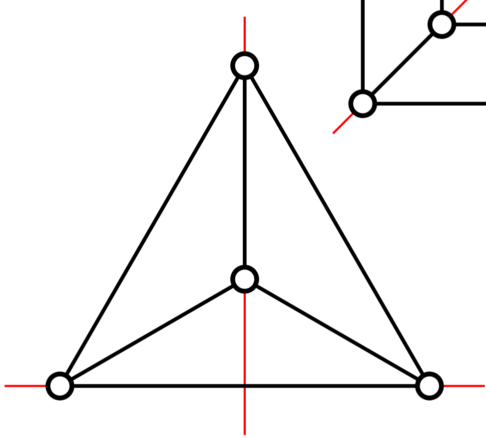
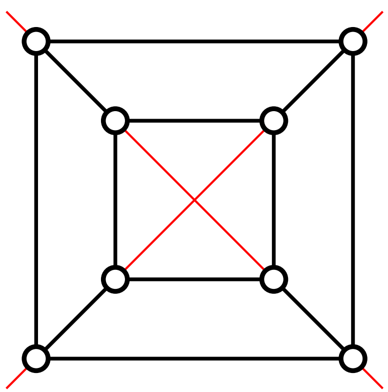
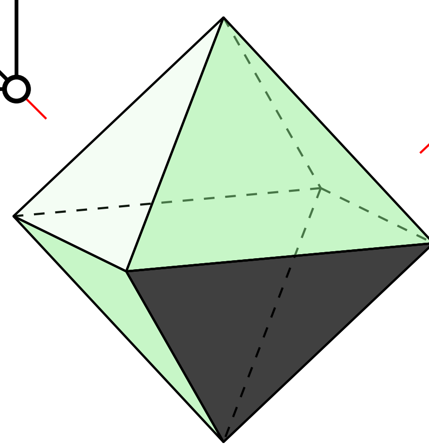
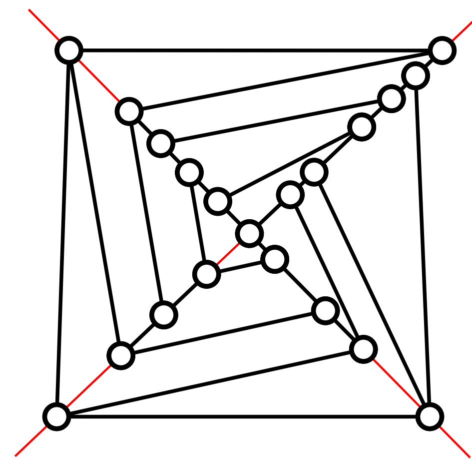
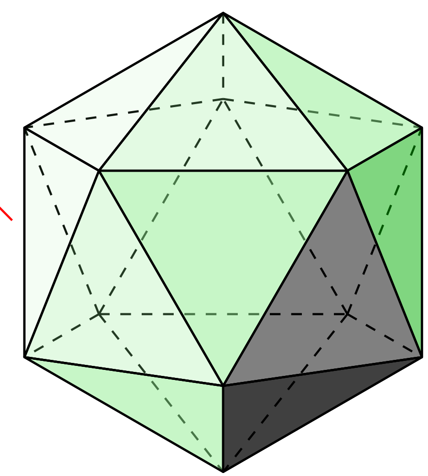

Line Cover Numbers of the Platonic Solids

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tetrahedron	4	6	4	6	6	2	
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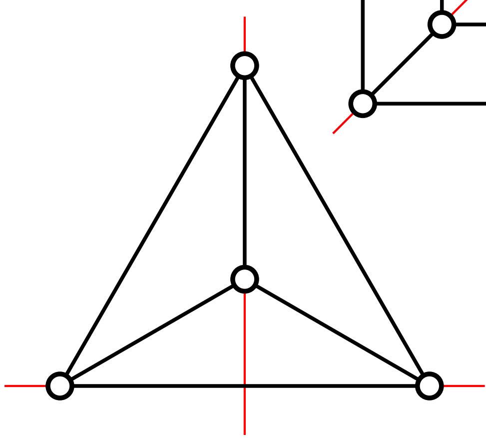
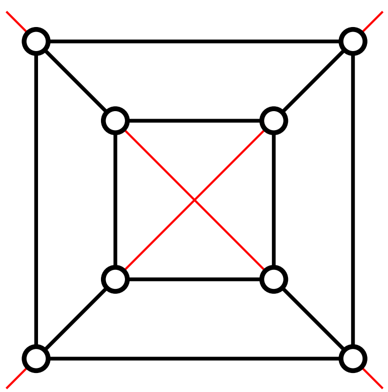
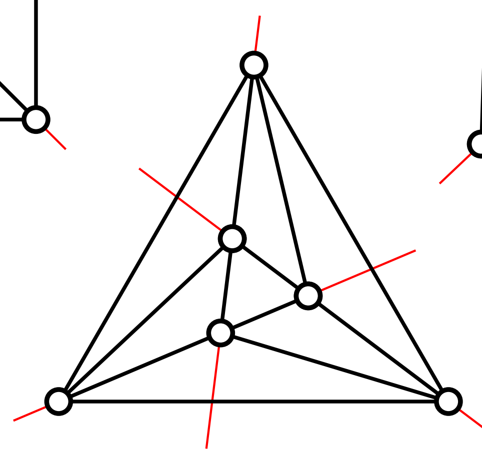
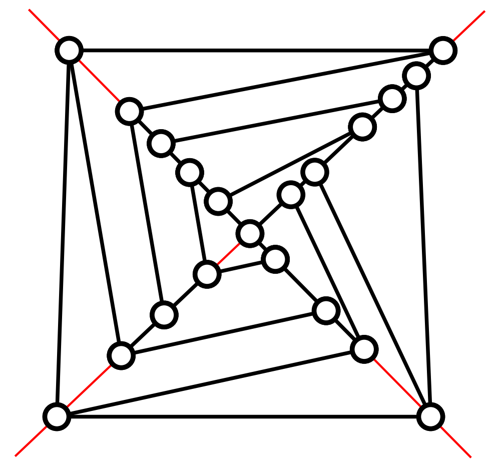
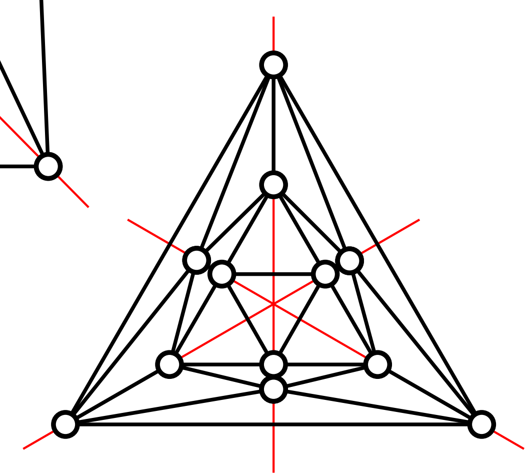
dodecahedron	20	30	12	9...10	9...10	2	
icosahedron	12	30	20	13...15	13...15	3	

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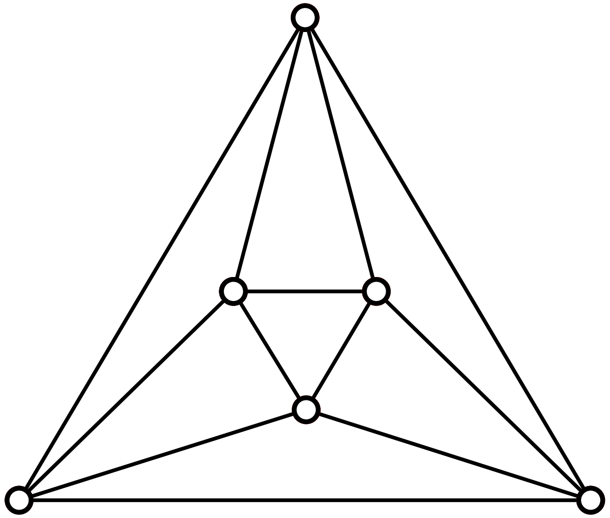

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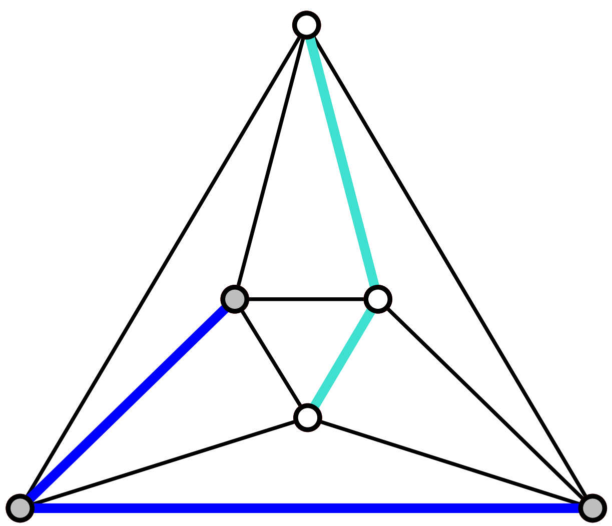

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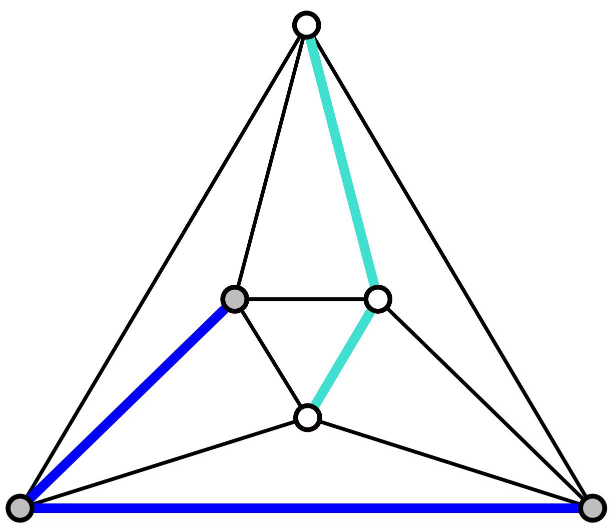

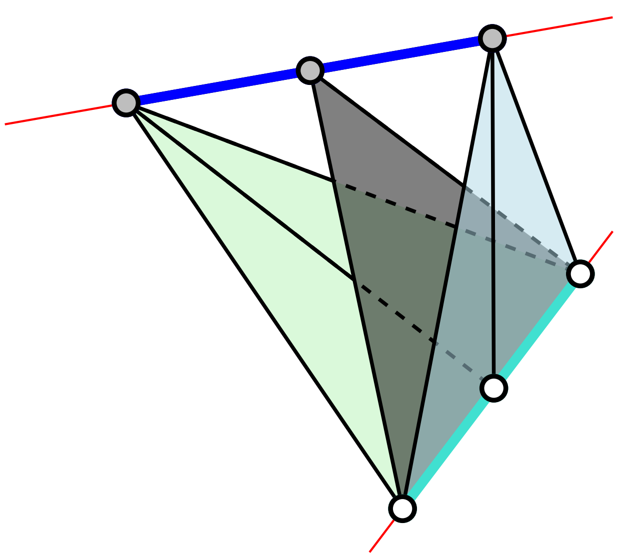

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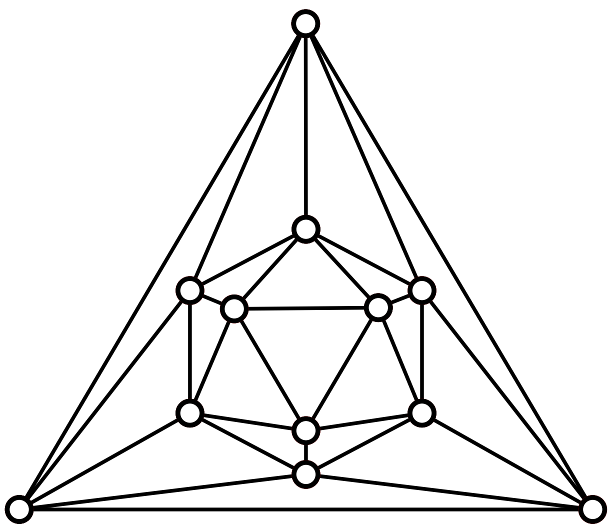

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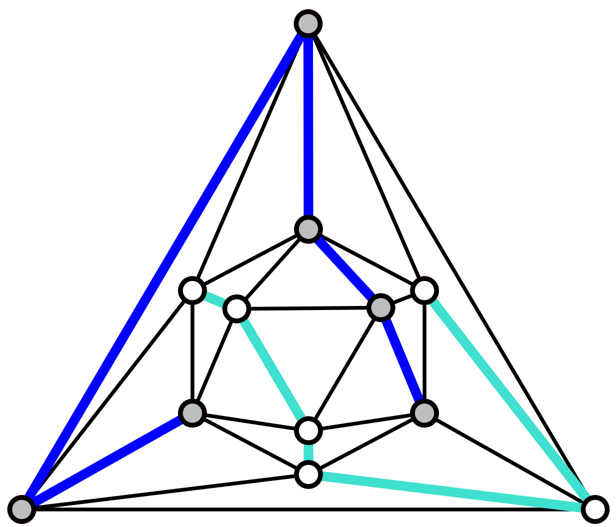

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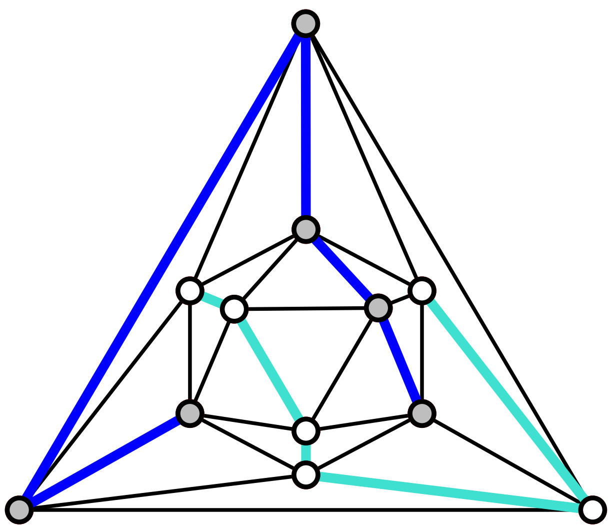

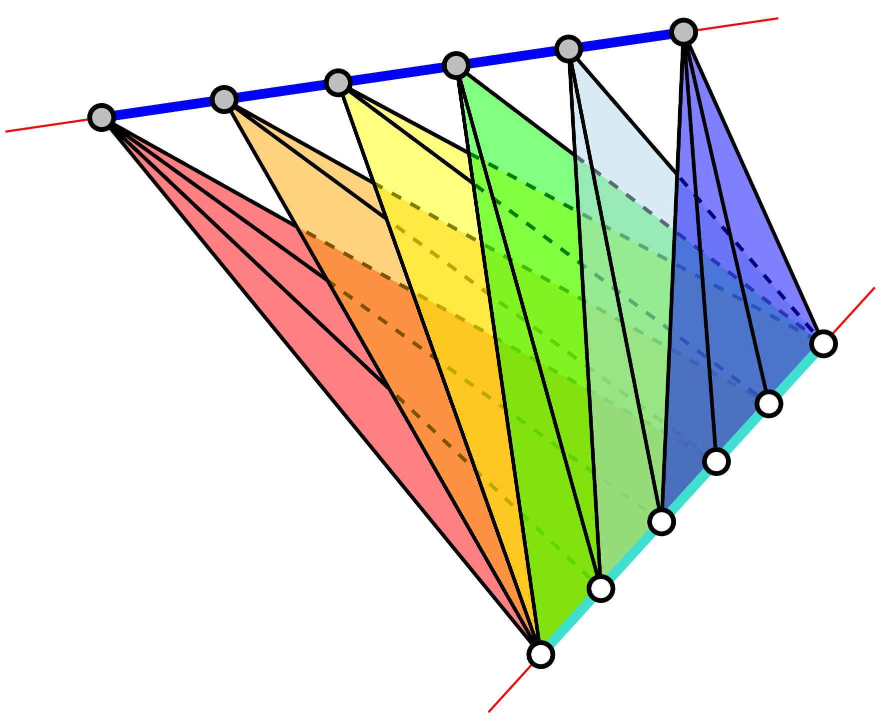

Graphs vs. Sets

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- Graphs are defined by a set of edges, which are sets of two elements.

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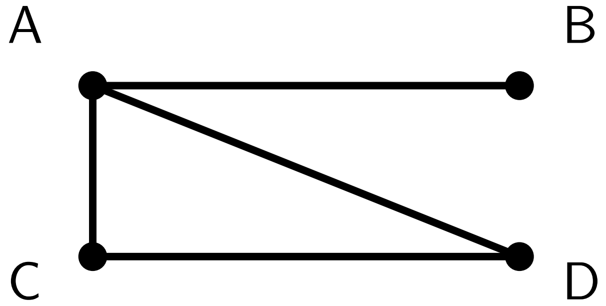

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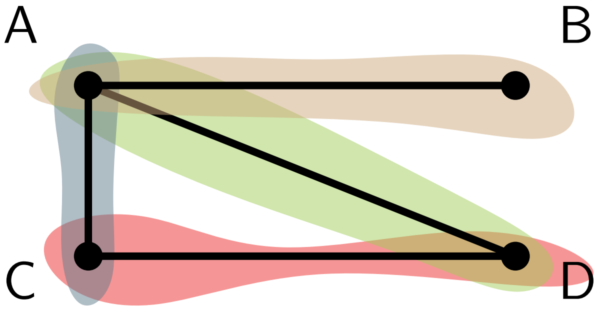

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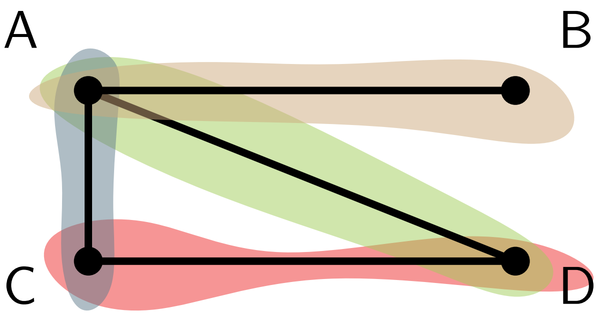

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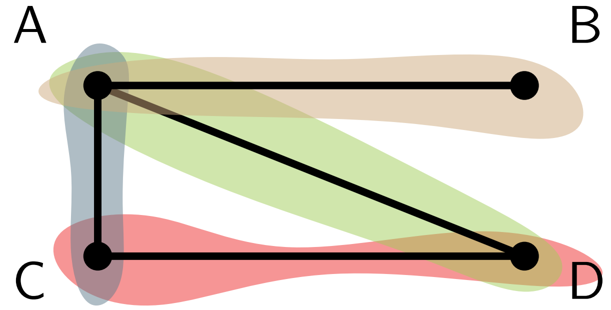

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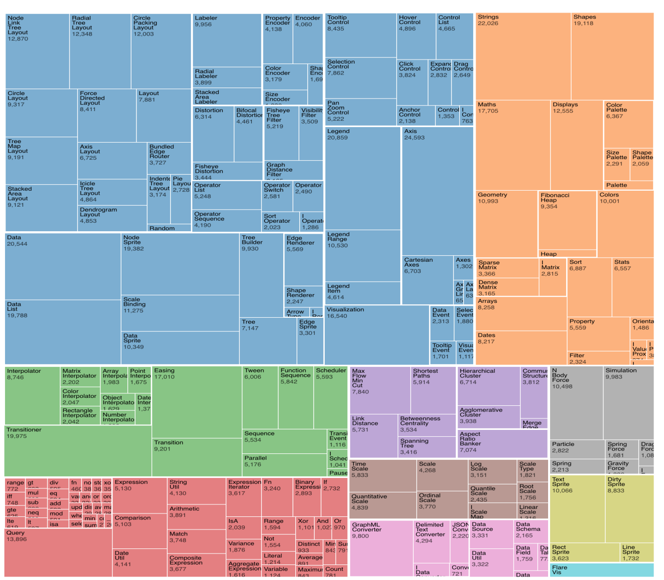

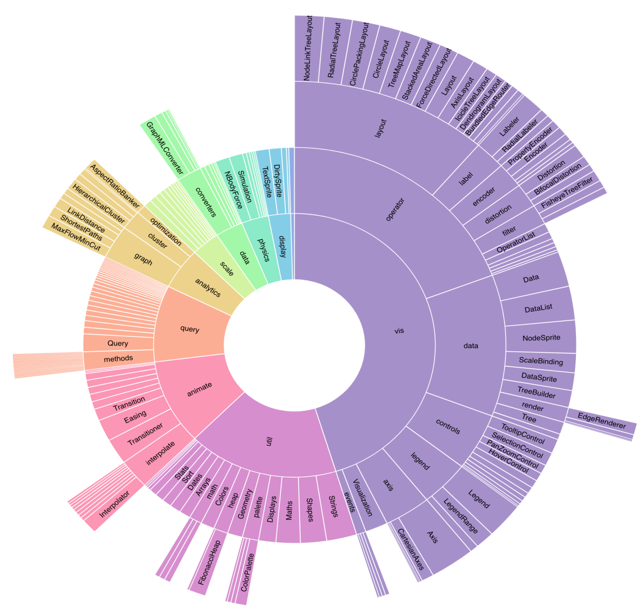

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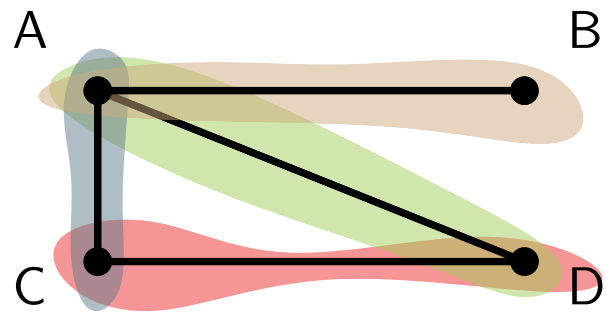

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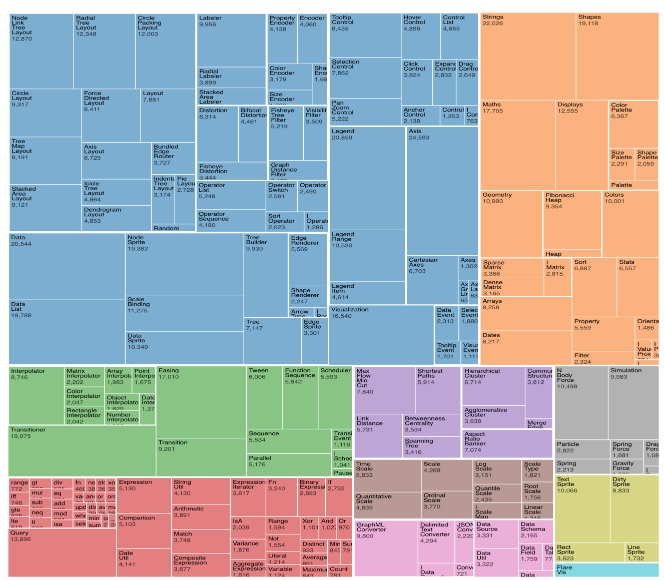

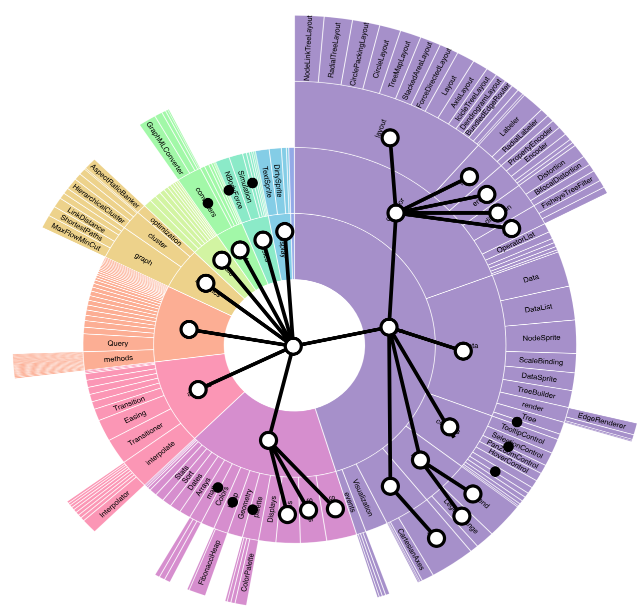

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Example:

$V = \{$black, red, green, yellow, blue, white, orange$\}$

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$V =$ {black, red, green, yellow, blue, white, orange}

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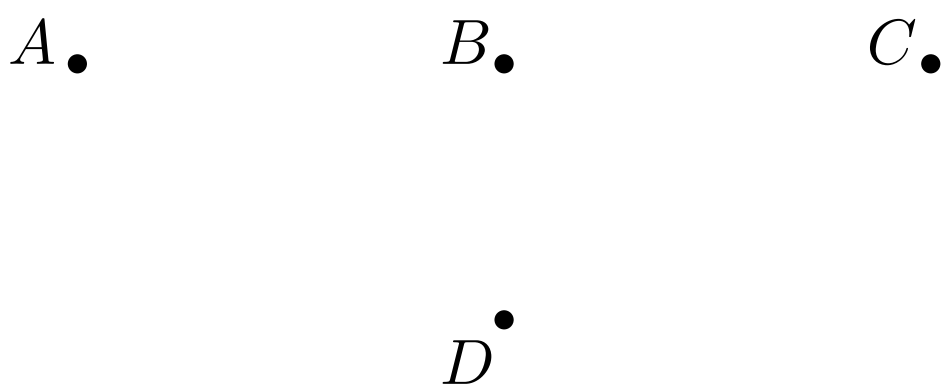

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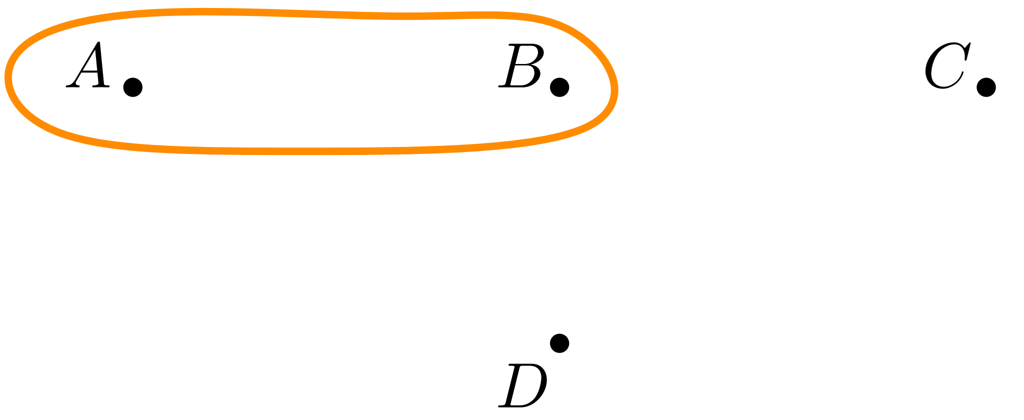

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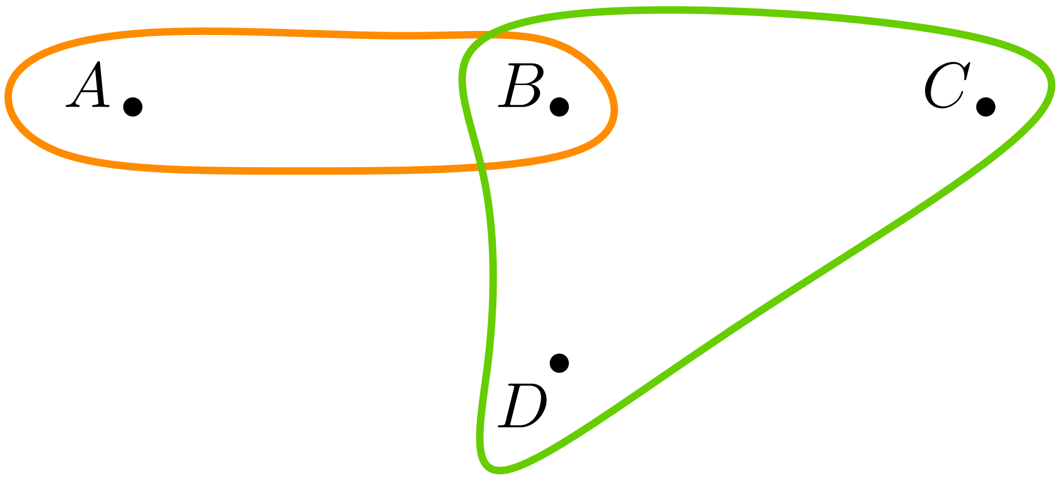

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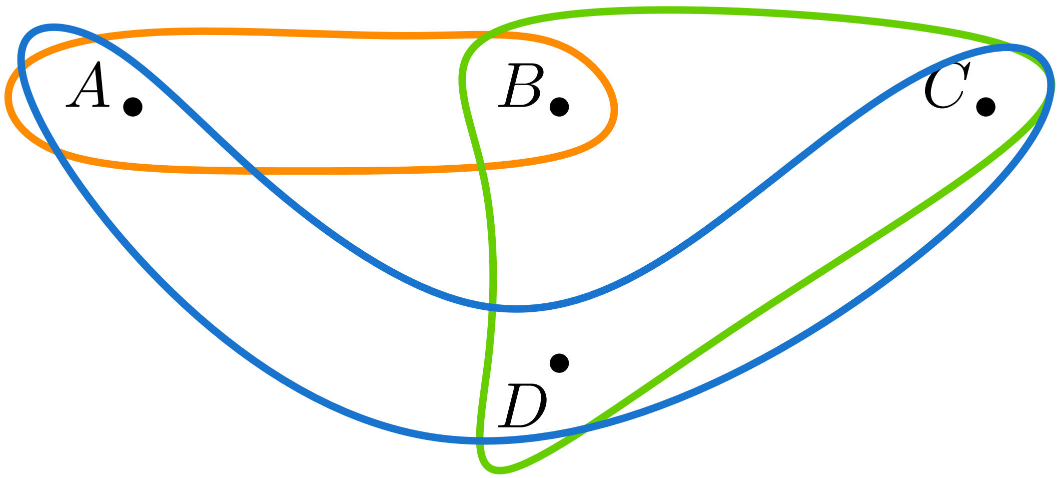

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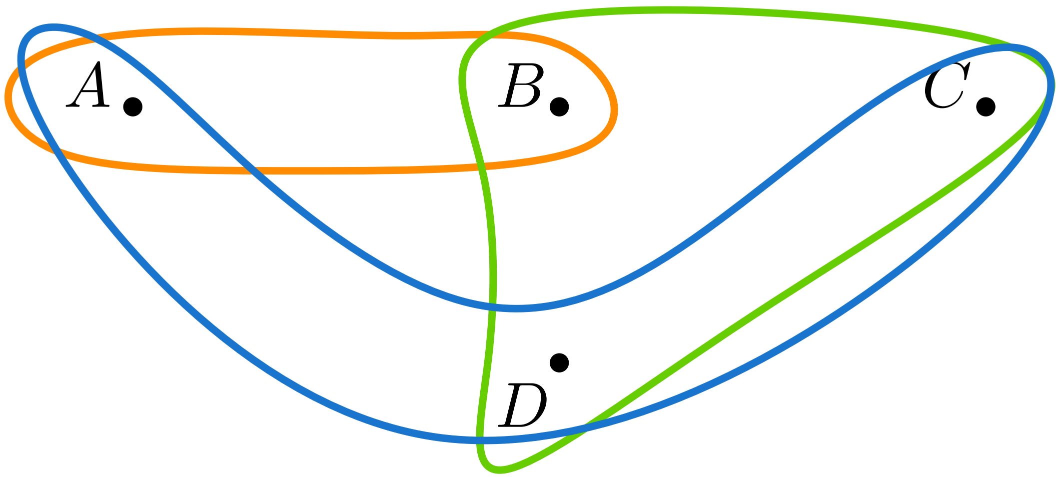

- spring embedder algorithm by Bertault and Eades 2000

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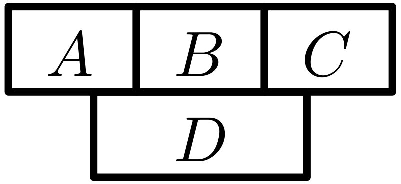

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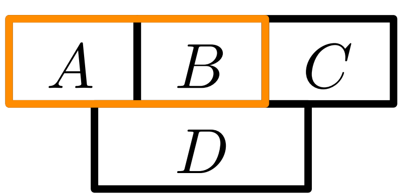

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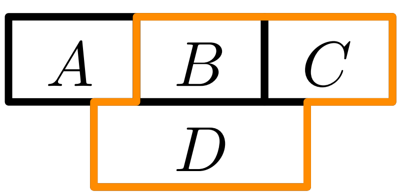

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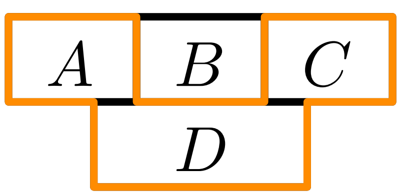

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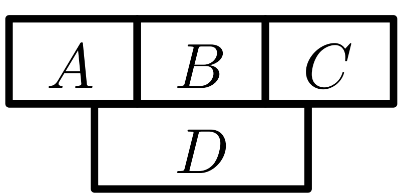

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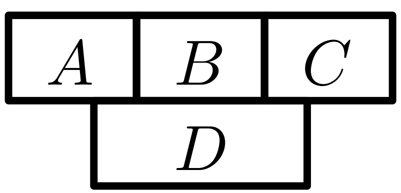

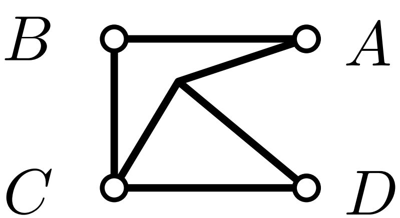

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Hypergraph drawing cont.

- Subset-based method gets easily confusing.
- **Alternatives:** subdivision-based & edge-based

Subdivision-based:
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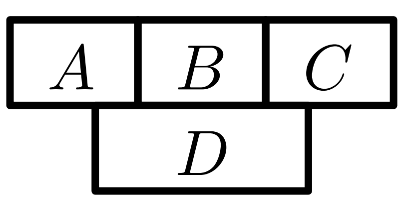

Edge-based:
- drawn as node-link diagram, with vertices as some nodes
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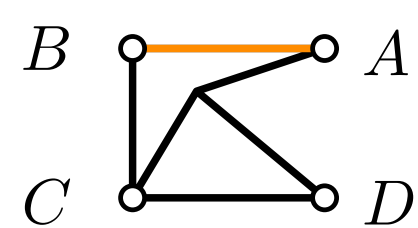
- … and more criteria



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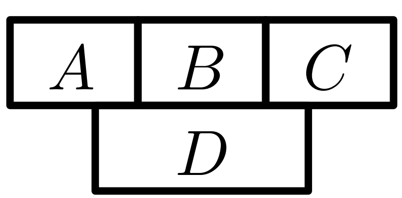

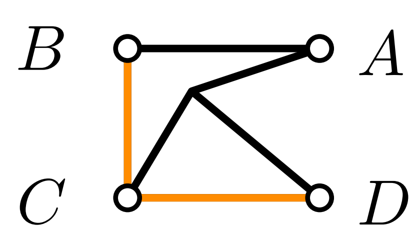

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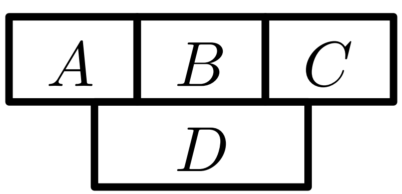

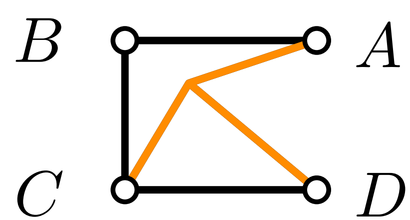

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Subdivision-based methods

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Concrete Euler Diagrams

Subdivision-based methods

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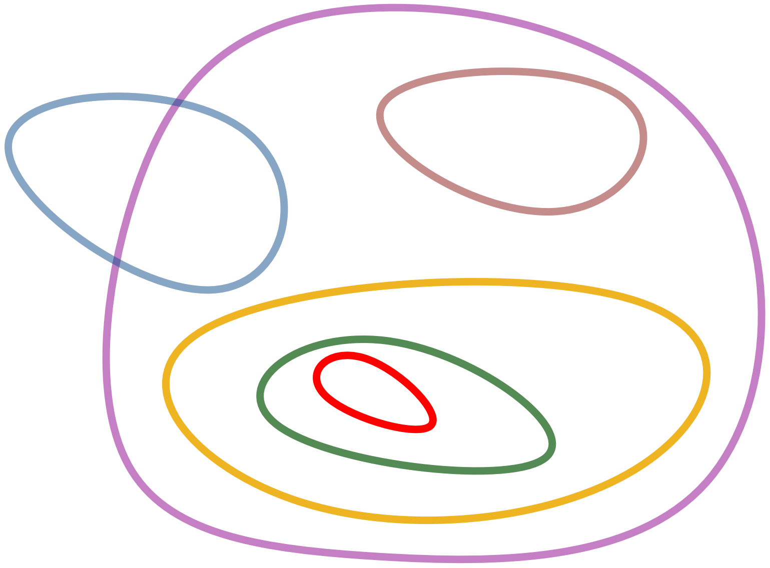

Subdivision-based methods

Concrete Euler Diagrams

- hyperedges are drawn as simple closed curves (interior/exterior)
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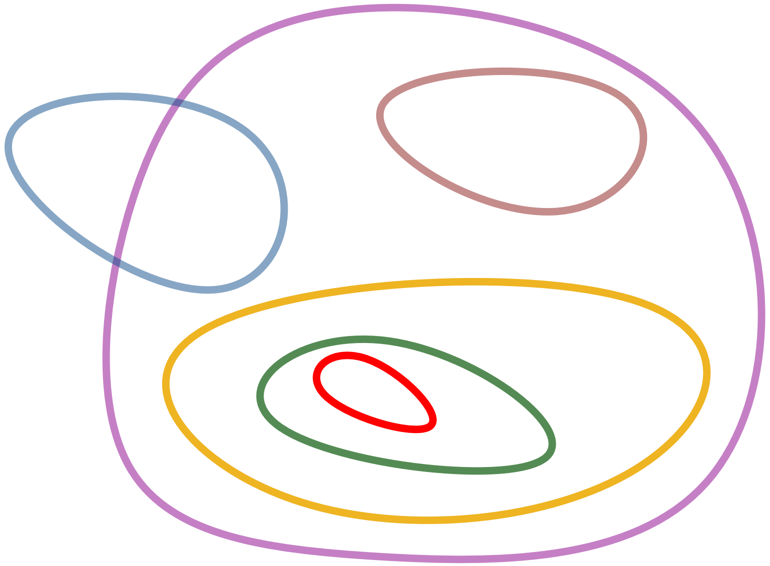

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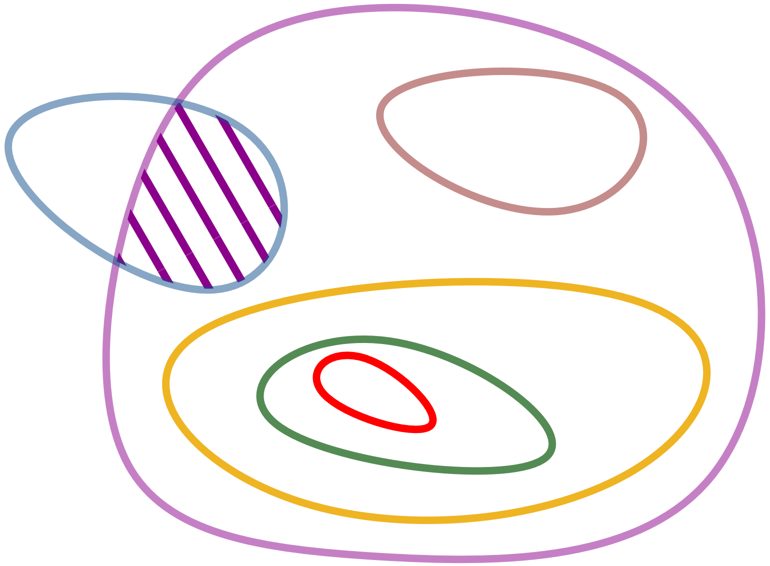

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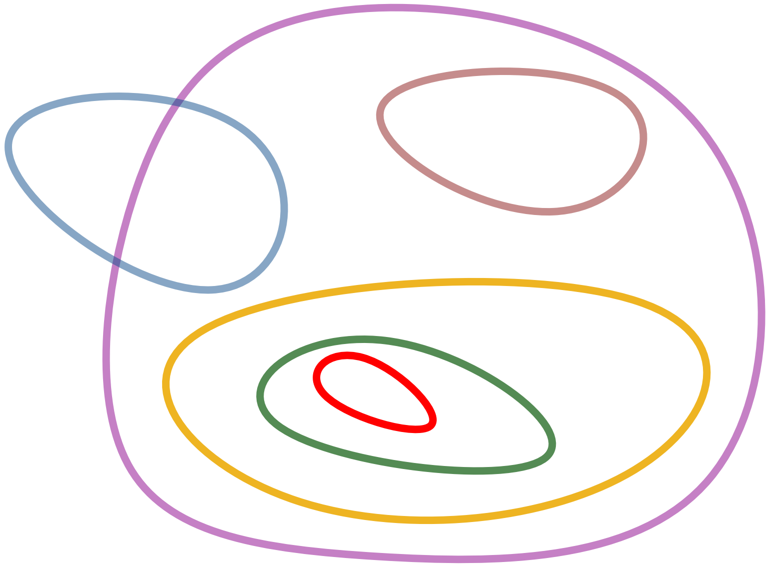

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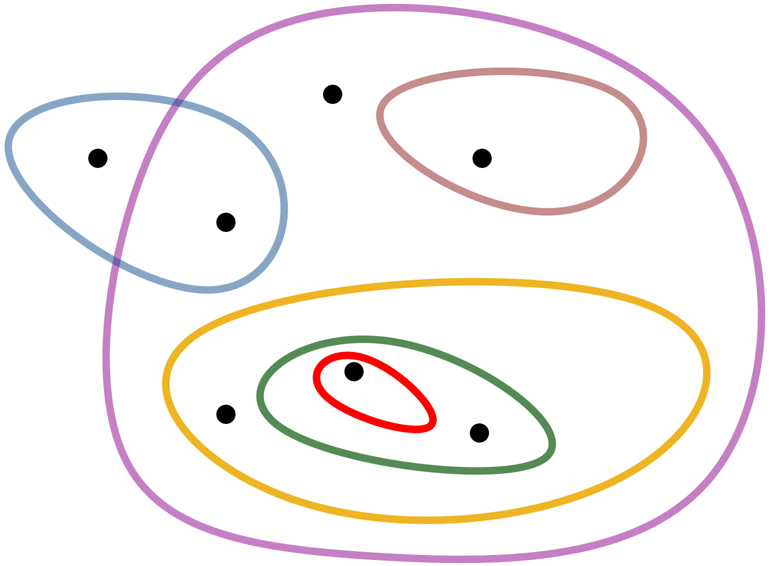

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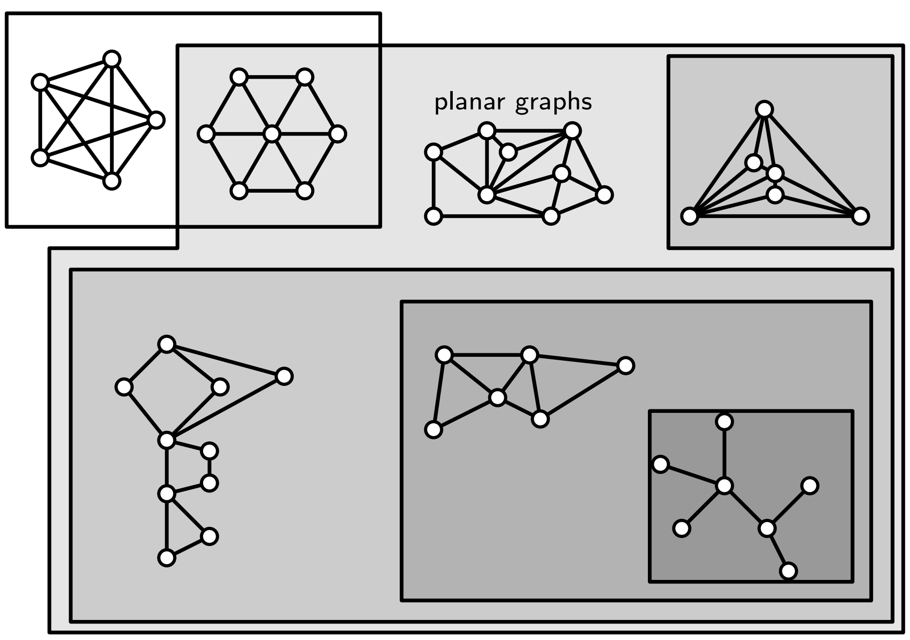

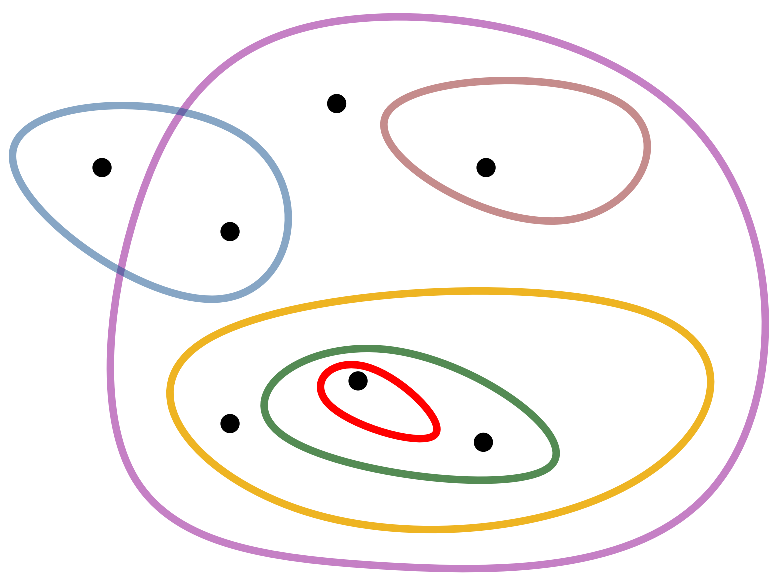

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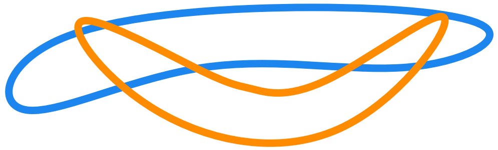
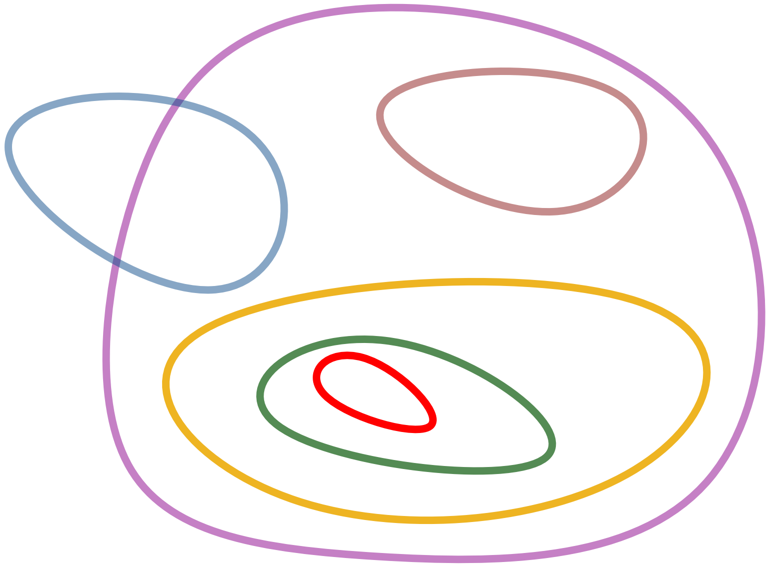


NO



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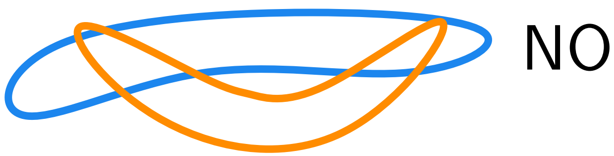
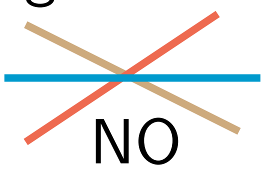
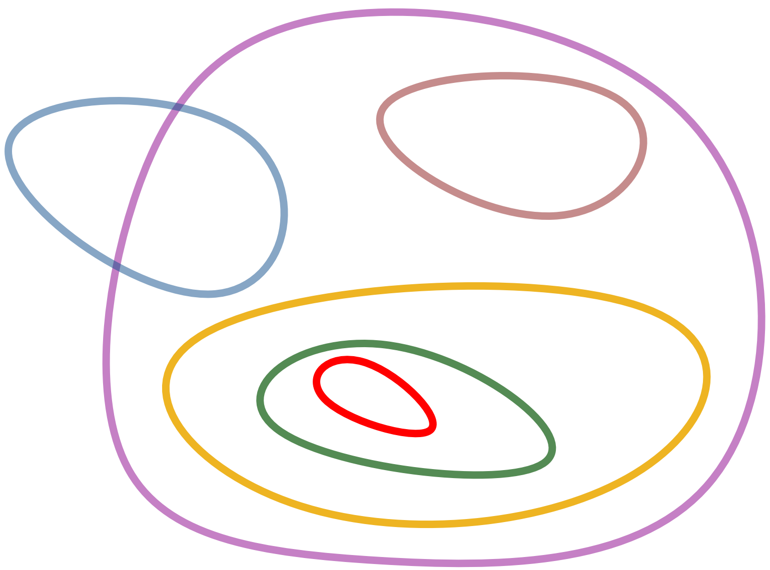


- only proper crossings





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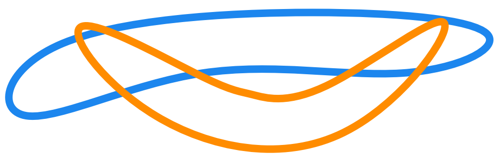

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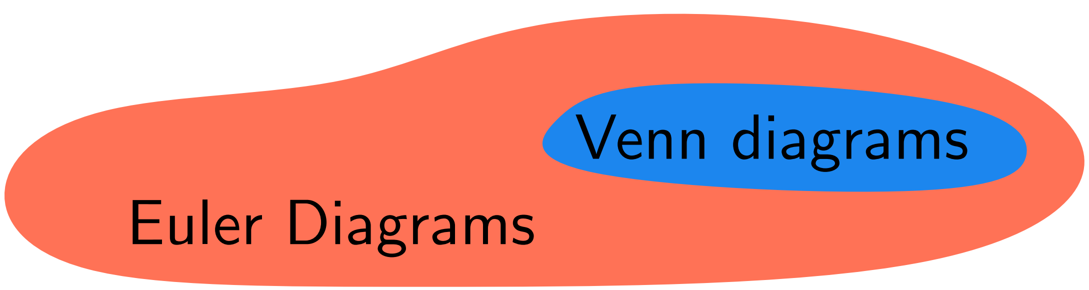

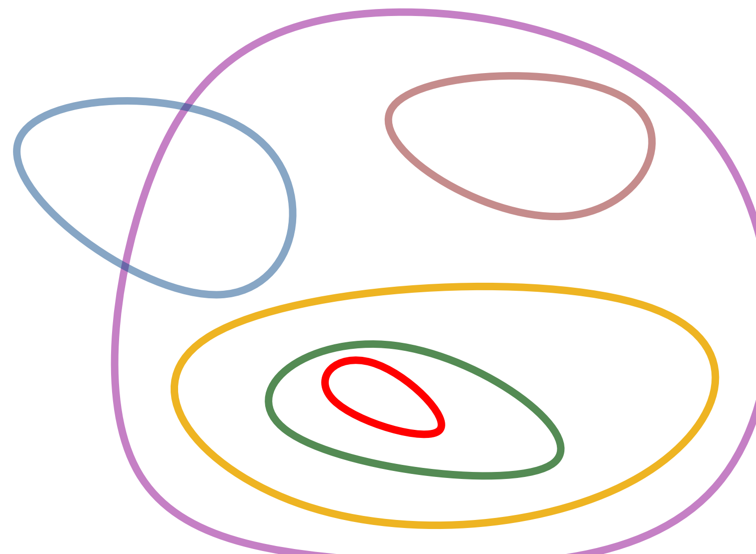

Edge-based methods

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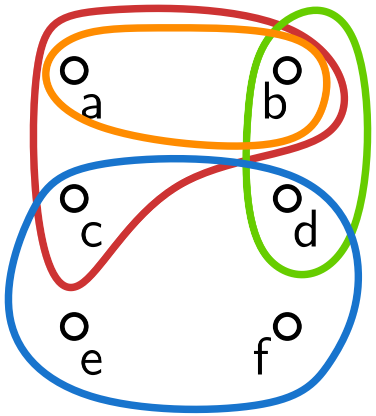
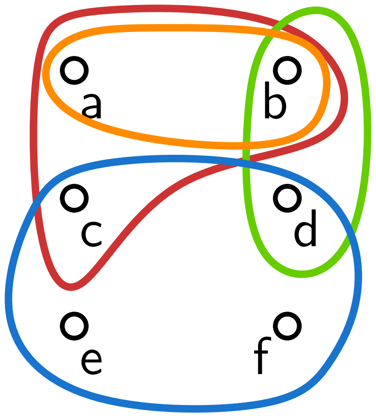


subset-based
drawing

Edge-based methods

Examples:



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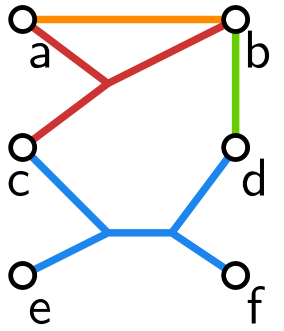

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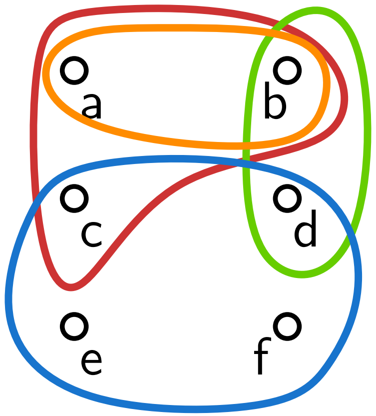

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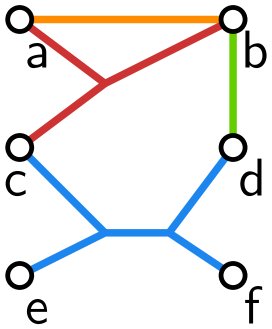

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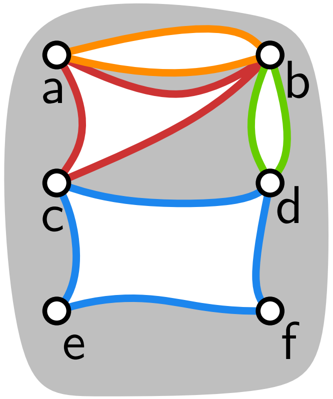

Zykov representation

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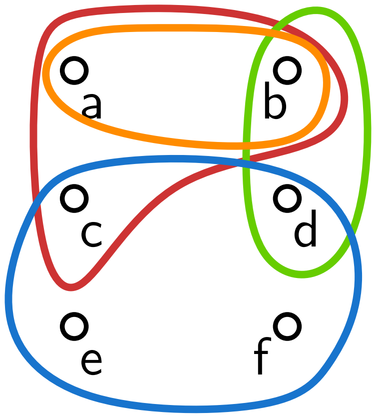

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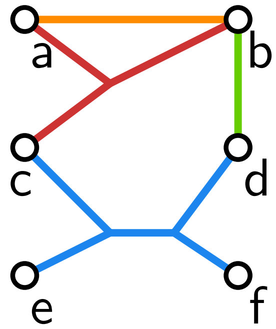

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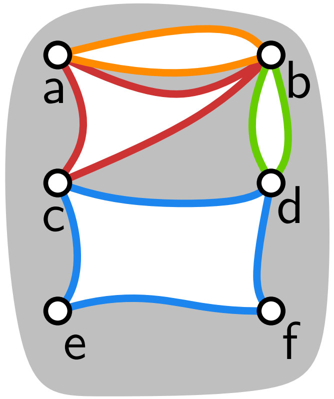

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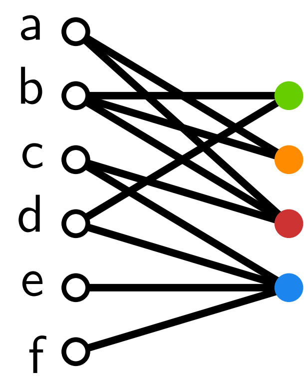

incidence representation

Edge-based methods

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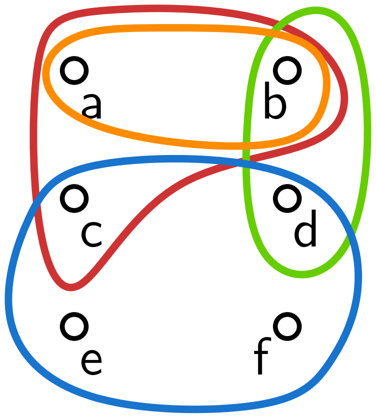

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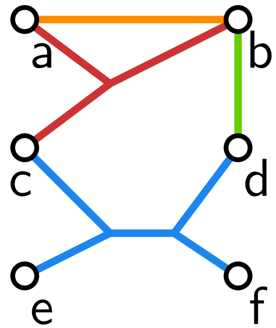

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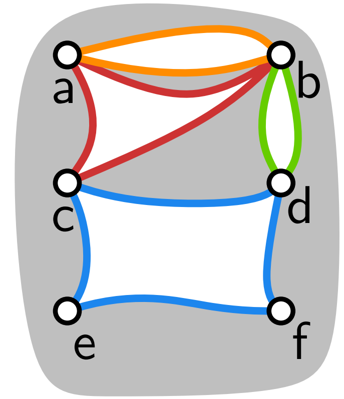

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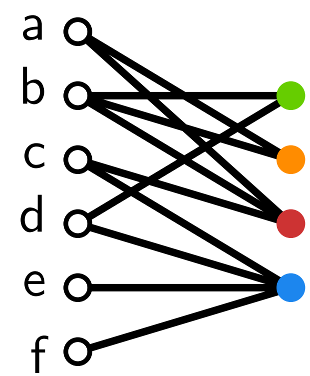

incidence representation

planarity is equivalent in all three models

Edge-based methods

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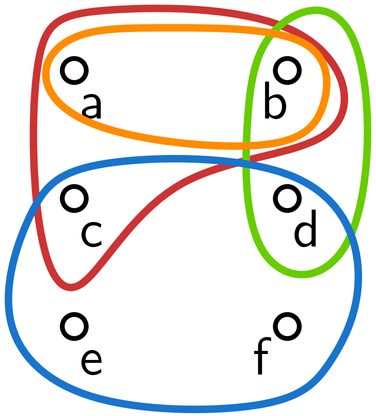

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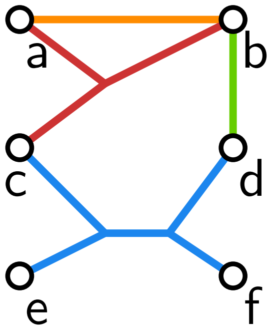

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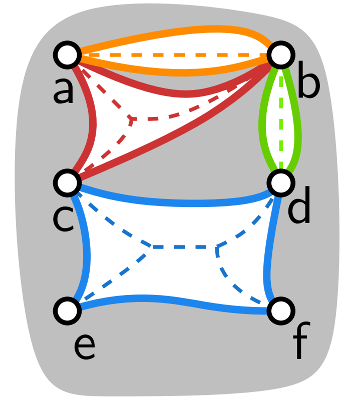

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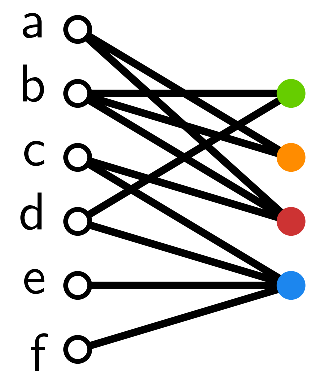

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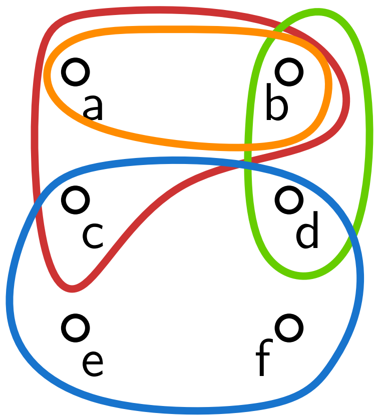

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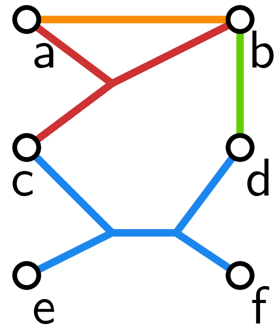

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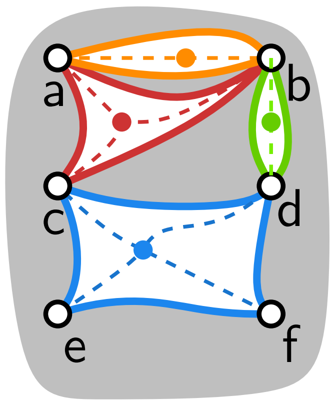

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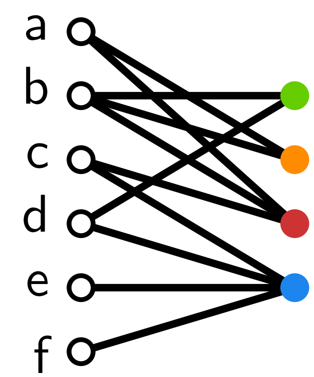

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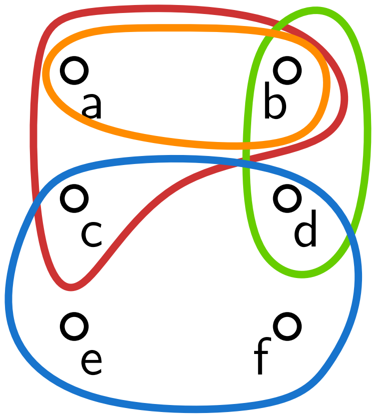

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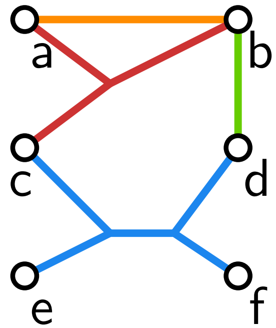

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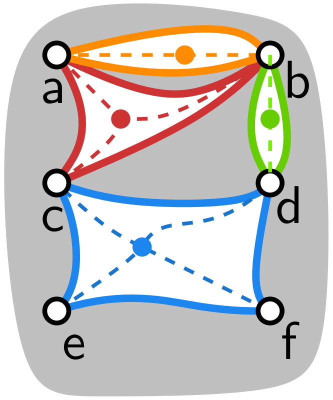

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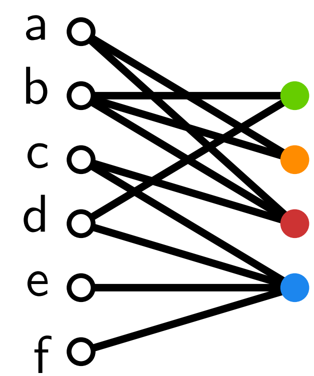

incidence representation

“planarity” NP-complete

planarity is equivalent in all three models $\in$ P

Edge-based methods

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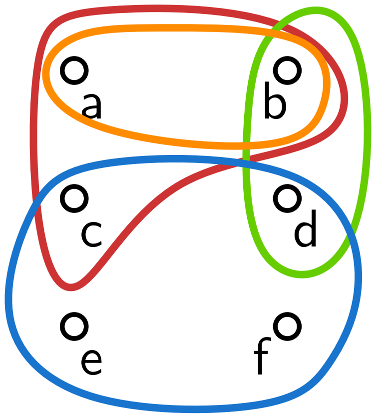

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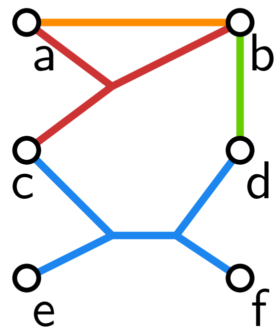

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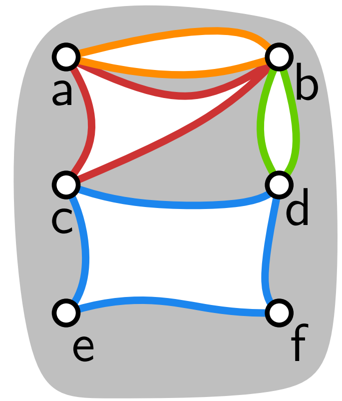

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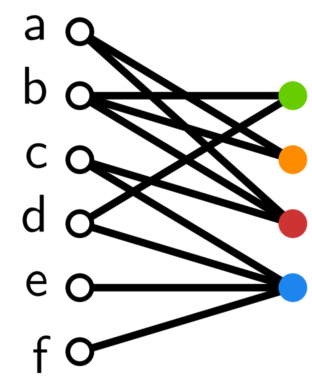
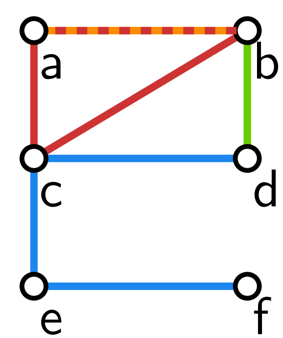


incidence representation

“planarity” NP-complete

planarity is equivalent in all three models $\in$ P

Def.: Support of a hypergraph is a graph such that every hyperedge induces a connected subgraph.



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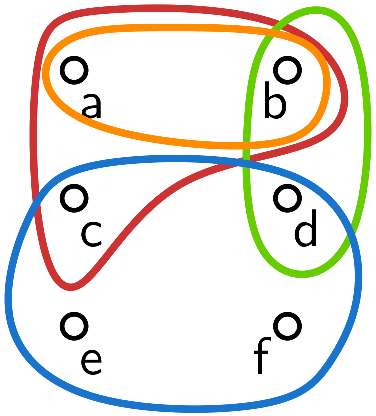

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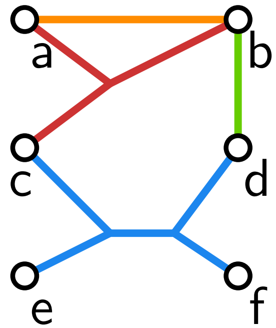

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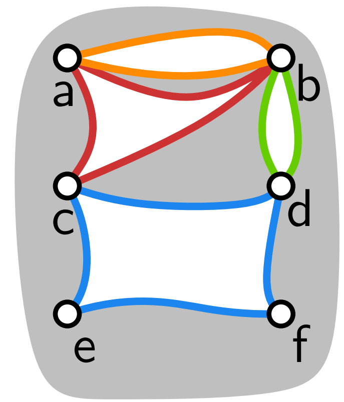

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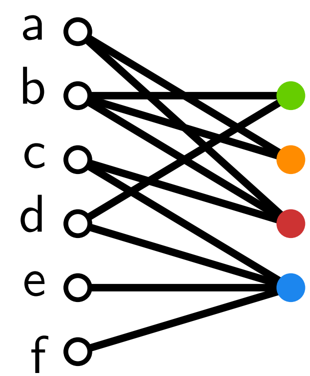

incidence representation

planarity is equivalent in all three models $\in$ P

"planarity"
NP-complete

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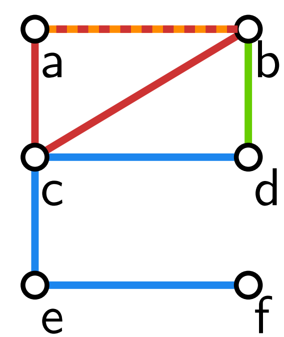
= planar support



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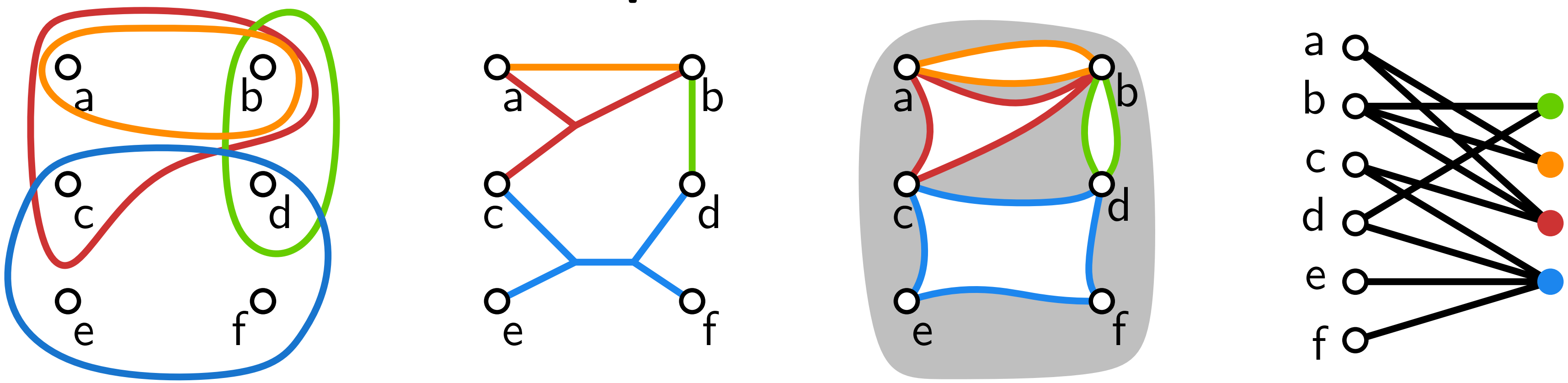

subset-based drawing | edge-based drawing | Zykov representation | incidence representation

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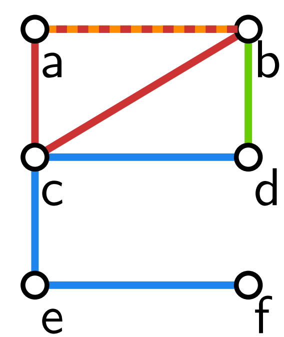

Test for cycle-, tree-, or cactus-support is feasible.

Simultaneous embeddings of hypergraphs

Simultaneous embeddings of hypergraphs

- Athenstädt et al. [GD'14] studied embeddings of two *partitions*:

Simultaneous embeddings of hypergraphs

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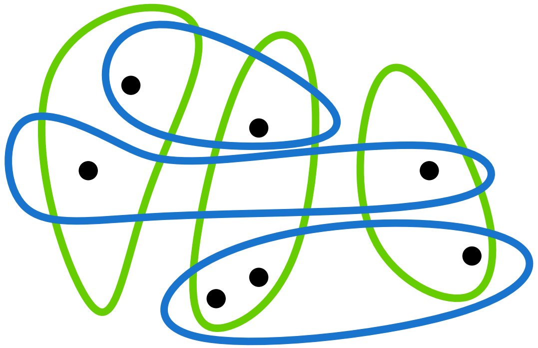
  3 Models



weak embedding
(no crossing sets)

Simultaneous embeddings of hypergraphs

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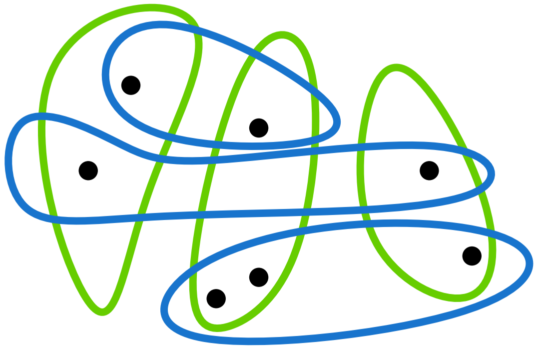
  3 Models



weak embedding
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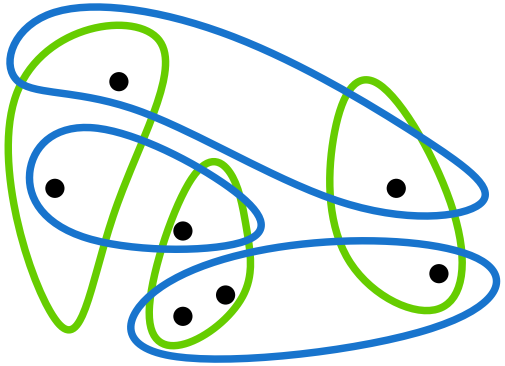

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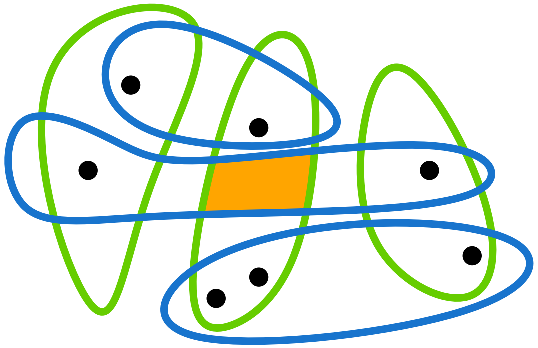

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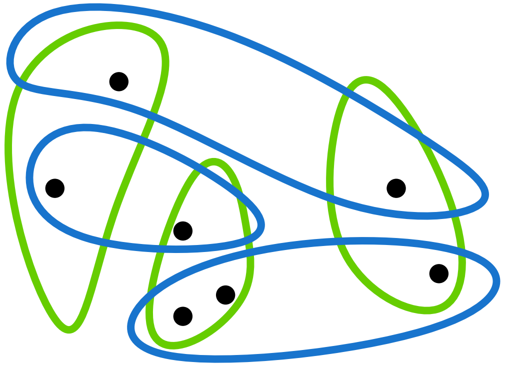

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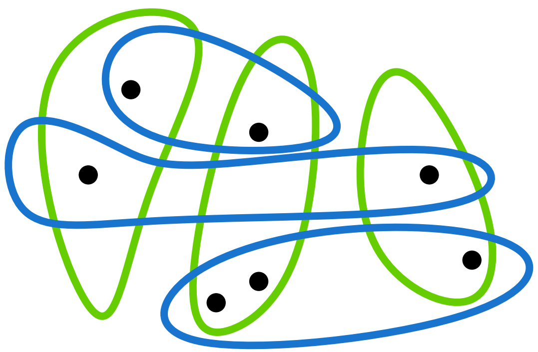

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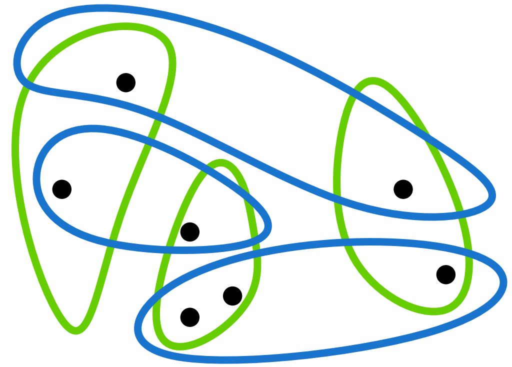

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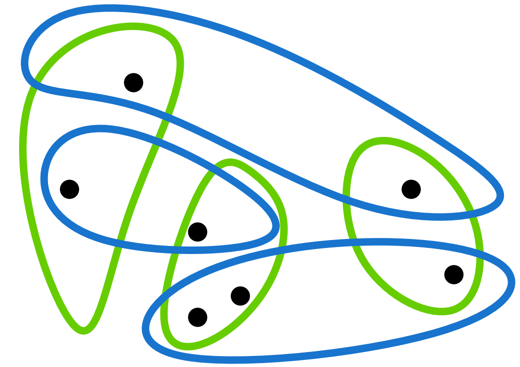

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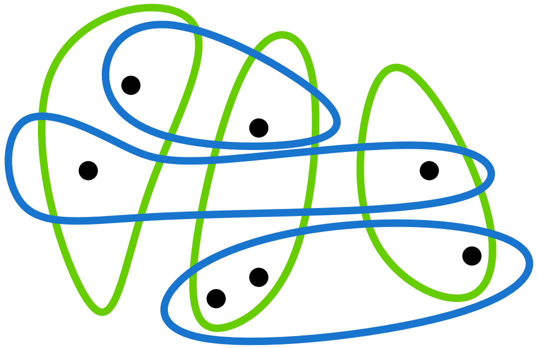

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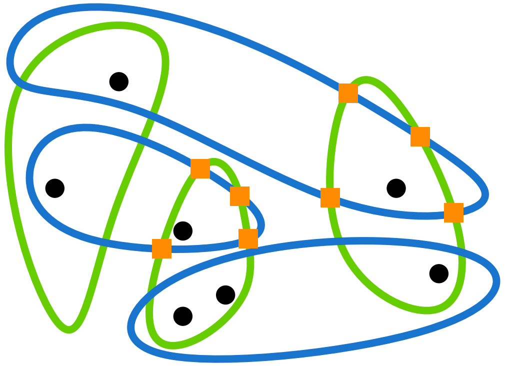

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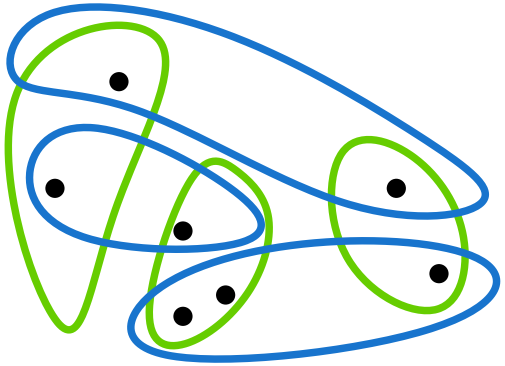

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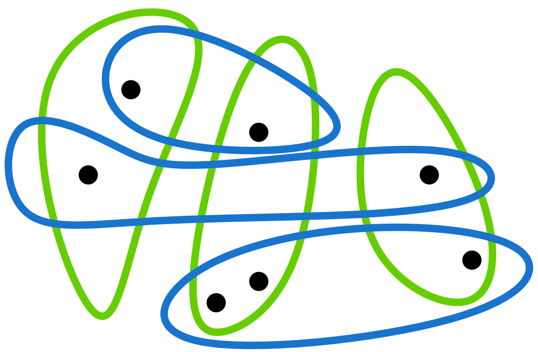

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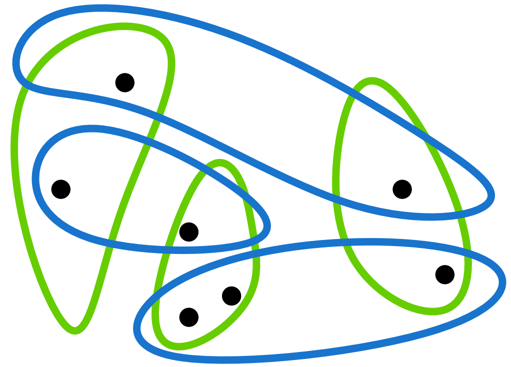
↑

always exists



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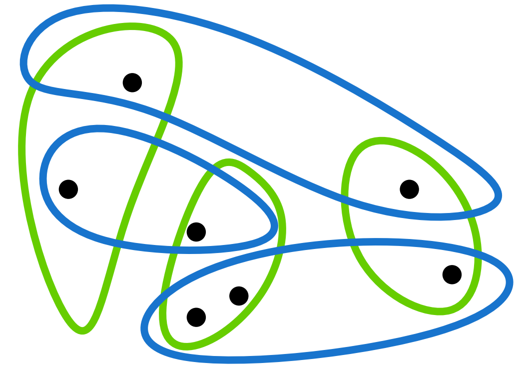

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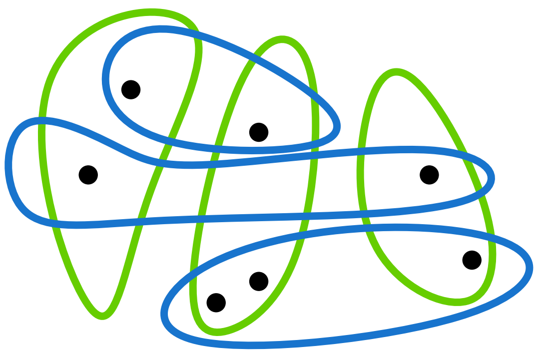

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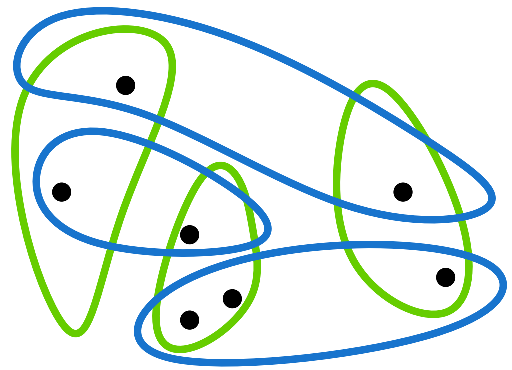

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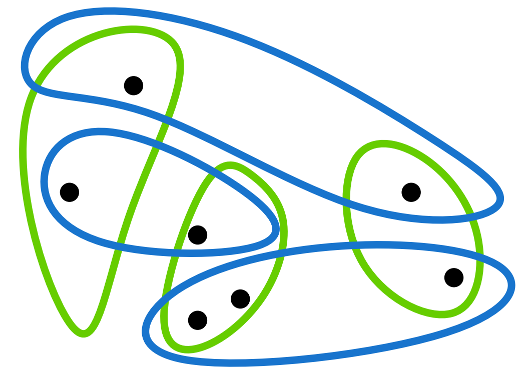

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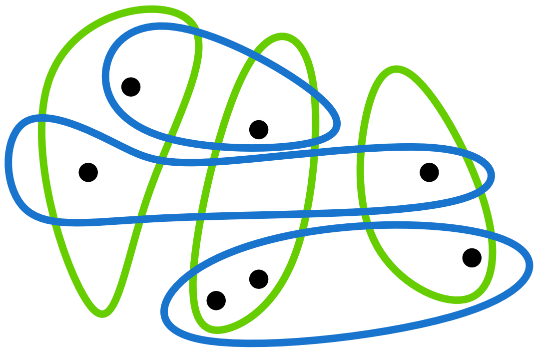	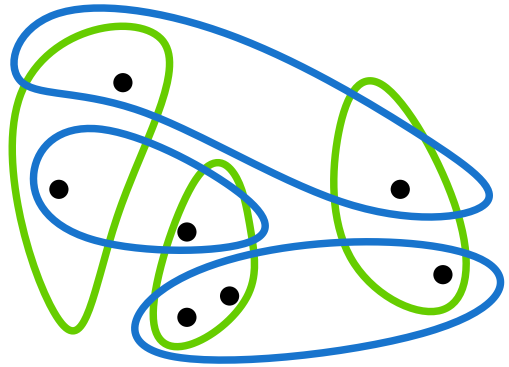	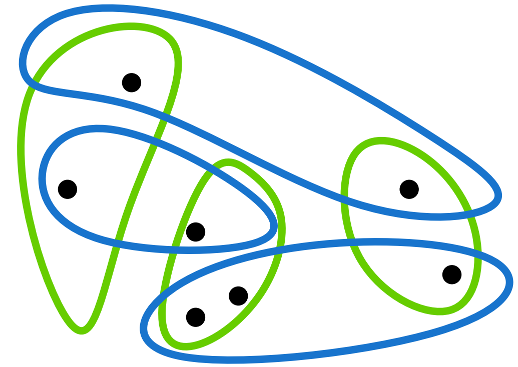
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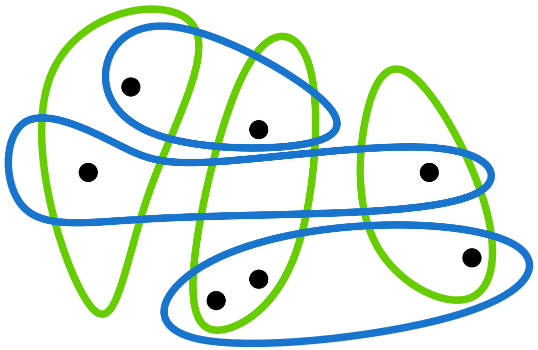	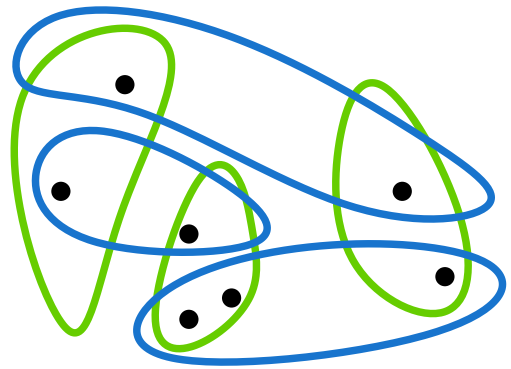	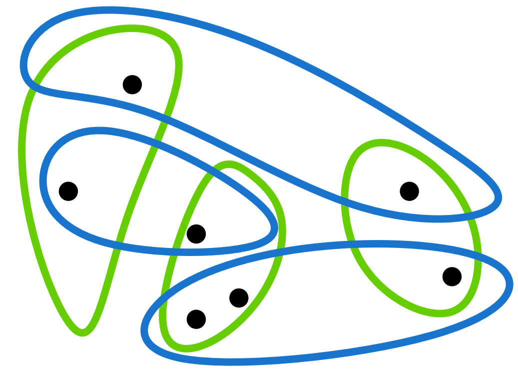
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- No results for more than 2 partitions or more general hypergraphs.

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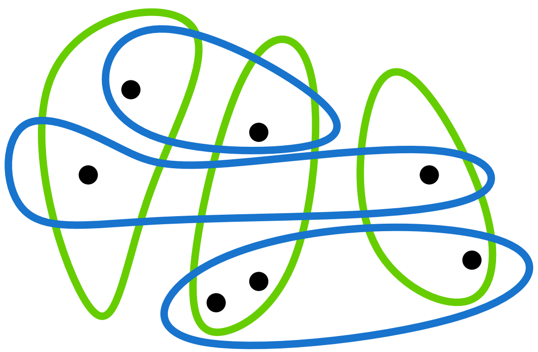

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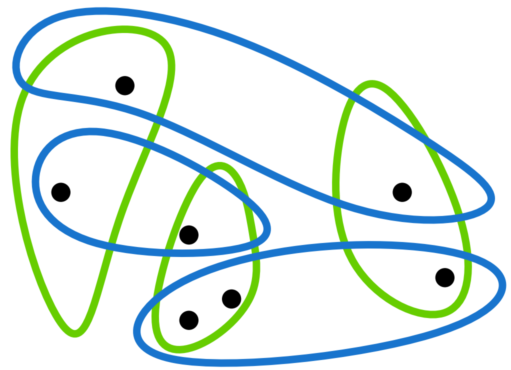

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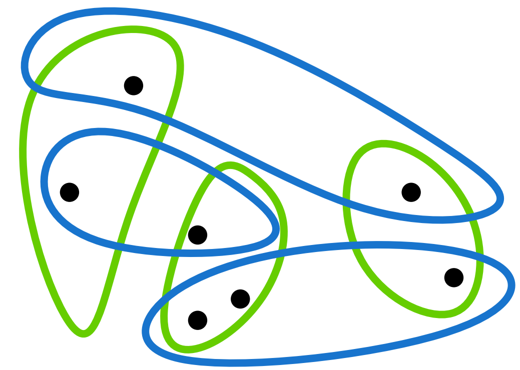

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can be tested efficiently

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  E.g., *linear* hypergraphs, where 2 hyperedges share at most 1 vertex.

Final words

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  - labeling
  - beyond planar graphs
  - right-angle-crossing drawings
  - universal point sets
  - topological drawings
  - representation as contact/intersection graphs
  - 3d graph drawing
  - layered drawings
  - bus drawings
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Thank you.